AN UNPUBLISHED THEOREM OF KRONECKER RESPECTING NUMERICAL EQUATIONS.

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1. In the winter semester of 1890–91, when he lectured on algebra for the last time, Kronecker developed a theory of the algebraic equation with numerical coefficients to which he attached a great deal of importance but which unhappily he did not live to publish. Professor Hensel, of Marburg, Kronecker's literary legatee and editor of his works, possesses notes of these lectures which he kindly placed in my hands last summer with the very generous permission to make any use of them I might wish. In reading these notes, I came upon the theory to which I have referred, and it seemed to me so characteristic of Kronecker and so full of interest for the algebraist that I have chosen it for the subject of this address.

All who have read Kronecker's later writings are familiar with his contention that the theory of the algebraic equation in its final form must be based solely on the rational integer, algebraic numbers being excluded and only such relations and operations being admitted as can be expressed in finite terms by means of rational numbers and therefore ultimately by means of integers. These lectures of 1890–91 are chiefly concerned with the development of such a theory, and in particular with the proof of two theorems which therein take the place of the fundamental theorem of algebra as commonly stated.

The first of these theorems relates to equations whose coefficients are rational functions of one or more variables and is characterised by Kronecker as the fundamental theorem of algebra in the algebraic sense. It is:

For every given algebraic equation

$$F(x) = x^n - c_1x^{n-1} + \cdots + (-1)^n c_n = 0$$

there exists a congruence of the form

$$F(x) \equiv (x - x_1)(x - x_2) \cdots (x - x_n) \qquad (\mathrm{mod} d f_i - c_i, g - c),$$

where the x_i's are variables, the f_i's are the elementary symmetric functions of the x_i's, c belongs to the domain of rationality of the coefficients c_i, and g is a rational, integral, but in general non-symmetric function of the x_i's whose form depends on the algebraic character of $F(x)$, and is such that the modular system whose elements are the $f_i - c_i$ and $g - c$ is prime.

Here, as always in Kronecker's algebraic writings, "exists" means "can be actually found by a limited number of rational operations." I shall not attempt to sketch the proof of this theorem since that would necessitate giving an account of a large part of Kronecker's arithmetical theory of algebraic functions. Moreover the theorem is fully discussed in the paper entitled "Ein Fundamentalsatz der allgemeinen Arithmetik" (*Journal für die reine und angewandte Mathematik*, volume 100). I have stated it mainly to give some indication of the method—the use of indeterminates, such as the x_i's, and modular systems—by which Kronecker makes the introduction of the algebraic numbers unnecessary when the isolating of conjugate numbers is not required. That it is unnecessary in the latter case also Kronecker shows in his paper "Ueber den Zahlbegriff" (*Journal für die reine und angewandte Mathematik*, volume 101) and in the demonstration which follows.

Kronecker's second theorem, "the fundamental theorem of algebra in the arithmetical sense," relates to equations with rational numerical coefficients. It is:

Every rational integral function $f(x)$ of the nth degree with rational numerical coefficients can, by a variation of the coefficients which is definite in character but as small as one may please, be reduced to a product of ν linear and $(n - \nu)/2$ quadratic factors, all with rational coefficients, ν being an integer determined by the coefficients of $f(x)$.

As hardly need be said, to establish this theorem in a manner meeting the requirements of the theory of which it forms a part, one must, without going beyond the domain of the rational numbers and therefore without at any point assuming the existence of linear factors of $f(x)$ or of roots of $f(x) = 0$, derive a general method by which in the case of any given function the requisite variation of the coefficients may be obtained directly from the coefficients themselves. I shall give Kronecker's proof modified in certain particulars for convenience of exposition.

2. Since it is evidently only necessary to prove the theorem

for the case in which $f(x)$ has no multiple factors or linear factors with rational coefficients, we shall suppose $f(x)$ to be without such factors. Then $f(x) = 0$ has no rational roots, and therefore, from Kronecker's standpoint, no roots. But it may be the case that, δ representing a positive number taken as small as we please, there exist one or more x-intervals (x', x'') such that

$$f(x')\, f(x'') < 0 \text{ and } |f(x)| < \delta \text{ for } x' \leqq x \leqq x''.$$

Any such interval we shall call a *zero interval* of $f(x)$, or, when convenient, a "root" of $f(x) = 0$.

Every interval (a, b) such that $f(a)\, f(b) < 0$ contains at least one zero interval of $f(x)$.

For if $a < x < x + h < b$, and g denote any number greater than all the values of $|f^{(r)}(x)/r!|$ $(r = 1, 2, \cdots, n)$ in (a, b), we shall have $|f(x + h) - f(x)| < \delta$ when $h < \delta/|g + \delta|$. Find an integer ν such that $(b - a)/\nu < \delta/(g + \delta)$, and divide (a, b) into ν equal sub-intervals. Among these, since $f(a)\, f(b) < 0$, there will be at least one, (a_ν, b_ν), such that $f(a_\nu)\, f(b_\nu) < 0$. Moreover if x denote any number in (a_ν, b_ν), both $|f(x) - f(a_\nu)| < \delta$ and $|f(b_\nu) - f(x)| < \delta$, and either $f(a_\nu)\, f(x) < 0$ or $f(x)\, f(b_\nu) < 0$, and therefore $|f(x)| < \delta$. Hence (a_ν, b_ν) is a zero interval of $f(x)$.

If throughout an interval (a, b), $|f'(x)| > c$, where c denotes some positive number, and if ξ denote any number in (a, b), then

$$\operatorname{sgn} f'(\xi) = \operatorname{sgn}[f(b) - f(a)].$$

That the sign of $f'(x)$ is constant in (a, b) follows from the preceding theorem. And if $a < x < x + h < b$, and g have the same meaning as above, then since $f(x + h) - f(x) = h[f'(x) + hf''(x)/2! + \cdots]$, we shall have $\operatorname{sgn}[f(x + h) - f(x)] = \operatorname{sgn} f'(x)$ when $|hf''(x)/2! + \cdots| < |f'(x)|$ and therefore when $h < c/(c + g)$. Hence, if (a, b) be divided into intervals (x_i, x_{i+1}) of the magnitude $c/(c + g)$ or less, we shall have $\operatorname{sgn}[f(x_{i+1}) - f(x_i)] = \operatorname{sgn} f'(\xi)$ for every value of i and therefore $\operatorname{sgn}[f(b) - f(a)] = \operatorname{sgn} f'(\xi)$.

From this theorem it immediately follows that if $|f'(x)| > c$ throughout (a, b), (a, b) will contain a sub-interval (x', x'') in which $|f(x)| < \delta$ only when $f(a)\, f(b) < 0$, and then but one such sub-interval. It also follows that:

Between two consecutive zero intervals of $f(x)$, (x_1', x_1'') and

(x_2', x_2''), in both of which $|f'(x)| > c$, there is a zero interval of $f'(x)$.

For $\operatorname{sgn} f'(x_1'') = \operatorname{sgn}[f(x_1'') - f(x_1')] = \operatorname{sgn} f(x_1'') = \operatorname{sgn} f(x_2') = -\operatorname{sgn}[f(x_2'') - f(x_2')] = -\operatorname{sgn} f'(x_2')$.

It is therefore evident that we shall be able to isolate the zero intervals of $f(x)$ if we can determine the size of an interval in which there cannot be both a zero interval of $f(x)$ and one of $f'(x)$. As will be shown, this can be found from the identity $D = P(x)\ f(x) + Q(x)\ f'(x)$ connecting $f(x)$, $f'(x)$, and the discriminant D of $f(x)$. We therefore proceed to establish this identity by a method which does not assume the existence of linear factors of an integral function.

3. The necessary and sufficient condition that the functions

$$F(x) = \prod_{i=1}^{m}(x - x_i) = x^m + \sum_{r=1}^{m}(-1)^r f_r x^{m-r}$$

and

$$G(x) = \prod_{k=1}^{n}(x - y_k) = x^n + \sum_{r=1}^{n}(-1)^r g_r x^{n-r}$$

have a common factor is the vanishing of the rational integral function of the coefficients f_r, g_r, defined by

$$\operatorname{Res}(F, G) = \prod_{ik}(x_i - y_k) = \prod_{i=1}^{m} G(x_i) = (-1)^{mn}\prod_{k=1}^{n} F(y_k) = R(f_r, g_r).$$

The roots of the equation $F(x)\ G(x) = 0$ being $x_1, \cdots, x_m$, $y_1, \cdots, y_n$, and the expressions for $R(f_r, g_r)$ in terms of these roots being $\prod_{i=}^{m} G(x_i)$, $(-1)^{mn}\prod_{k=1}^{n} F(y_k)$, we have, by the interpolation formula of Lagrange,

$$R(f_r, g_r) = G(x)\sum_{i=1}^{m}\frac{F(x)}{x - x_i}\frac{\prod_{i=1}^{m} G(x_i)}{G(x_i)F'(x_i)} + (-1)^{mn}F(x)\sum_{k=1}^{n}\frac{G(x)}{x - y_k}\frac{\prod_{k=i}^{n} F(y_k)}{F(y_k)G'(y_k)}.$$

Evidently the coefficient of $G(x)$ in this identity can be reduced

to the form $\Sigma_{i=1}^{m}[a_0(x_i)x^{m-1} + \cdots + a_{m-1}(x_i)]/F'(x_i)$, where $a_0(x_i), \cdots, a_{m-1}(x_i)$ are integral functions of the f_r, g_r and of x_i, of degree not higher than $m-1$ in x_i. But, by Euler's formulas, $\Sigma_{i=1}^{m} x_i^{\lambda}/F'(x_i) = 1$ or 0 according as $\lambda = m-1$ or $< m-1$. Therefore the coefficient of $G(x)$ is a polynomial $\varphi(x)$, of degree $m-1$ at most, with coefficients which are integral functions of the f_r, g_r. The coefficient of $F(x)$ is reducible to a polynomial $\psi(x)$ of like character of degree $n-1$ at most. Therefore

$$R(f_r, g_r) = \varphi(x)\, G(x) + \psi(x)\, F(x), \tag{1}$$

where $\varphi(x)$ and $\psi(x)$ have the character just indicated.

Let

$$F_1(x) = x^m + \sum_{r=1}^{m} (-1)^r c_r x^{m-r}$$

and

$$G_1(x) = x^n + \sum_{r=1}^{n} (-1)^r d_r x^{n-r}$$

be functions with any given rational coefficients and which therefore are in general not resolvable into linear factors. We are to prove that the vanishing of $R(c_r, d_r)$ is the necessary and sufficient condition that $F_1(x)$ and $G_1(x)$ have a common factor.

If in (1) we set $f_r = c_r$, $g_r = d_r$, we obtain an identity of the form

$$R(c_r, d_r) = \varphi_1(x)G_1(x) + \psi_1(x)F_1(x), \tag{2}$$

from which it immediately follows that, if $F_1(x)$ and $G_1(x)$ have a common factor, $R(c_r, d_r)$ must be 0. And we can show as follows that if $F_1(x)$ and $G_1(x)$ have no common factor, $R(c_r, d_r)$ cannot be 0.

For if $F_1(x)$ and $G_1(x)$ have no common factor, we can by means of the euclidean algorithm derive an identity of the form $1 = P_1(x)F_1(x) + Q_1(x)G_1(x)$, where $P_1(x)$ and $Q_1(x)$ are integral functions of x. This identity may be written

$$1 = F(x)P_1(x) + G(x)Q_1(x) + (F_1(x) - F(x))\, P_1(x) + (G_1(x) - G(x))Q_1(x),$$

from which it follows that

$$1 \equiv F(x)P_1(x) + G(x)Q_1(x) \pmod{f_r - c_r, g_r - d_r}.$$

Setting $x = x_1, x_2, \cdots, x_m$, successively, and taking the product of the resulting congruences, we have

$$1 \equiv \Pi G(x_i)\Pi Q_1(x_i) \equiv R(f_r, g_r)\cdot\Gamma(f_r) \pmod{f_r - c_r, g_r - d_r},$$

where $\Gamma(f_r)$ is an integral function of the f_r's. Therefore finally, setting $f_r = c_r$, $g_r = d_r$, we have

$$1 = R(c_r, d_r)\cdot\Gamma(c_r),$$

from which it follows, since $\Gamma(c_r)$ cannot be infinite, that $R(c_r, d_r)$ is not 0.

In the following discussion, by the resultant of two functions of the form $f(x) = c_0x^m - c_1x^{m-1} + \cdots + (-1)^m c_m$ and $g(x) = d_0x^n - d_1x^{n-1} + \cdots + (-1)^n d_n$, we shall mean the number $R = c_0^n d_0^m R[c_r/c_0, d_r/d_0]$, and by the discriminant of $f(x)$ we shall mean $D = \mathrm{Res}(f, f')/c_0$. Evidently R and D are integral when the given coefficients are integral.

4. Let $f(x) = \Sigma_{i=0}^{n} a_i x^{n-i}$ and $\varphi(x) = \Sigma_{i=0}^{m} b_i x^{m-i}$ be two polynomials with integral coefficients and having no common factor, whose resultant R is therefore an integer different from 0. By means of the identity $R = P(x)f(x) + Q(x)\varphi(x)$ we can determine an integer s such that no interval of the magnitude $1/s$ will contain zero intervals of both $f(x)$ and $\varphi(x)$.

For if x_1, x_2 denote any two values of x, we have identically

$$(1)\quad R = P(x_1)\frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_1 - x_2) + P(x_1)f(x_2) + Q(x_1)\varphi(x_1).$$

Since $|a_0x^n| > |a_1x^{n-1} + \cdots|$ for $|x| > (|a_0| + a_g)/|a_0|$, where a_g denotes the greatest of the numbers $|a_i|$, and similarly $|b_0x^m| > |b_1x^{m-1} + \cdots|$ for $|x| > (|b_0| + b_g)/|b_0|$, all the zero intervals of $f(x)$ and $\varphi(x)$ lie in the interval $(-c, c)$, where c denotes the first integer greater than both the numbers $(|a_0| + a_g)/|a_0|$ and $(|b_0| + b_g)/|b_0|$.

Let K denote the greatest of the three numbers derived from

$$P(x)\frac{f(x+\sigma) - f(x)}{\sigma} = \psi(x, \sigma), \quad P(x), \quad \text{and} \quad Q(x)$$

by replacing the coefficients by their absolute values, x by c, and σ by 1. Then any integer s such that

$$(2)\qquad K < (s-1)|R|$$

will meet the requirement stated above. For if s satisfy (2),

and if also $-c < x_1, x_2 < c$ and $|x_1 - x_2| < 1$, it will follow from (1), by dividing throughout by $(s-1)|R|$, that

$$\frac{1}{s-1} < |x_1 - x_2| + |f(x_2)| + |\varphi(x_1)|$$

and therefore, if $|x_1 - x_2| < 1/s$, that

$$|f(x_2)| + |\varphi(x_1)| > \frac{1}{s(s-1)}. \tag{3}$$

But since x_1, x_2 are independent variables subject only to the conditions $-c < x_1, x_2 < c$ and $|x_1 - x_2| < 1/s$, it follows from (3) that no interval of the magnitude $1/s$ can contain zero intervals of both $f(x)$ and $\varphi(x)$. On the contrary, if it contains a zero interval of one of the functions, the other will remain numerically greater than $1/s(s-1)$ throughout the interval.

We are now in a position to isolate and count the zero intervals of any polynomial $f(x)$. For if in the preceding discussion we suppose $\varphi(x) = f'(x)$ and, having determined s, divide the interval $(-c, c)$ into equal sub-intervals of the magnitude $1/s$ it follows from § 2 that those only of these sub-intervals contain zero intervals of $f(x)$ at whose extremities $f(x)$ has opposite signs, and that each such sub-interval contains one and but one zero interval of $f(x)$. Hence the number of such sub-intervals is the number of the zero intervals of $f(x)$. And since sgn $f(-c) =$ sgn $f(c)$ or sgn $f(-c) = -$ sgn $f(c)$ according as n, the degree of $f(x)$, is even or odd, the number ν of the zero intervals of $f(x)$ is even (or 0) or odd according as n is even or odd.

5. We are now equipped for proving that part of Kronecker's theorem which relates to the linear factors. For to the conditions used in determining s in § 4 let us add, for the case in which $\varphi(x) = f'(x)$ and therefore $R = D =$ Disc $f(x)$, the following:

$$\left|\frac{f(x+\sigma) - f(x)}{\sigma}\right| < (s-1)|D|, \quad \text{for } |x| < c \text{ and } |\sigma| \leqq 1.$$

Then, for x_1, x_2 such that $-c < x_1, x_2 < c$ and $|x_2 - x_1| < 1$, we have

$$|f(x_2) - f(x_1)| < (s-1)|D| \cdot |x_2 - x_1|. \tag{1}$$

Divide $(-c, c)$ into equal intervals of the magnitude $1/s$, and

of these intervals let (x', x'') be any one such that $f(x')\,f(x'') < 0$. Then, r denoting an integer which may be taken as great as we please, divide (x', x'') into $r\,|\,D\,|$ equal sub-intervals. Of these sub-intervals there will be one and but one at whose extremities $f(x)$ has opposite signs. If this sub-interval be (ξ', ξ''), it follows from (1) that, if $\xi' \leqq \xi \leqq \xi''$, then both $|\,f(\xi) - f(\xi')\,| < 1/r$ and $|\,f(\xi'') - f(\xi)\,| < 1/r$ and therefore, since either $f(\xi')\,f(\xi) < 0$ or $f(\xi)\,f(\xi'') < 0$, that $|\,f(\xi)\,| < 1/r$. Hence (ξ', ξ'') is a zero interval of $f(x)$ characterized by the inequalities

$$(2) \quad f(\xi')\,f(\xi'') < 0 \text{ and } |\,f(\xi)\,| < 1/r \text{ for } \xi' \leqq \xi \leqq \xi''.$$

There are ν such sub-intervals (§ 4). Represent them by

$$(3) \qquad (\xi_1', \xi_1''), (\xi_2', \xi_2''), \cdots (\xi_\nu', \xi_\nu'').$$

We are to prove that if ξ_i denote any number in (ξ_i', ξ_i''), we can reduce $f(x)$ to the form

$$(4) \quad f(x) = (x - \xi_1)(x - \xi_2) \cdots (x - \xi_\nu)Q(x) + R(x),$$

where $Q(x)$ is a polynomial without zero intervals and $R(x)$ a polynomial whose coefficients are as small as we please.

Represent $\Pi_{k=1}^{\nu}(x - \xi_k)$ by $\varphi(x)$. Then since $R(\xi_k) = f(\xi_k)$ for $k = 1, 2, \cdots, \nu$, we have, by the interpolation formula of Lagrange,

$$(5) \qquad R(x) = \sum_{k=1}^{\nu} \frac{f(\xi_k)}{\varphi'(\xi_k)} \frac{\varphi(x)}{x - \xi_k}.$$

But $\varphi'(\xi_k) = \Pi(\xi_k - \xi_i)$ $(i = 1, 2, \cdots, k-1, k+1, \cdots, \nu)$ and $|\,\xi_k - \xi_i\,| > 1/s$. For between the intervals (ξ_1', ξ_1'') and (ξ_2', ξ_2''), say, there lies at least one interval (η', η'') of magnitude $1/rs\,|\,D\,|$ which is a zero interval of $f'(x)$, and if η denote any number in (η', η''), since no interval of the magnitude $1/s$ can contain zero intervals of both $f(x)$ and $f'(x)$, we have $\eta - \xi_1 \geqq 1/s - 1/rs\,|\,D\,|$, $\xi_2 - \eta \geqq 1/s - 1/rs\,|\,D\,|$ and therefore $\xi_2 - \xi_1 \geqq 2/s - 2/rs\,|\,D\,|$. Hence $\xi_2 - \xi_1 > 1/s$ if, as we shall suppose, $r\,|\,D\,| \geqq 2$. Therefore

$$\left|\frac{f(\xi_k)}{\varphi'(\xi_k)}\right| < \frac{s^{\nu-1}}{r} \qquad (k = 1, 2, \cdots, \nu).$$

Again $\varphi(x)/(x - \xi_k)$ is a polynomial in x, of degree $\nu - 1$, whose coefficients after the first are less respectively than the numbers

$(\nu-1)c$, $(\nu-1)(\nu-2)c^2/2$, $\cdots$, $c^{\nu-1}$. Therefore, if k denote an integer greater than all these numbers, it follows from (5) that $R(x)$ can be reduced to the form

$$(6) \qquad R(x) = \frac{\nu s^{\nu-1}k}{r} S(x) = \frac{1}{r'} S(x), \text{ if } r = \nu s^{\nu-1} k r',$$

where the coefficients of $S(x)$ are all numerically less than 1, and r' is an arbitrary integer which may be taken as great as we please, great enough, therefore, to make the coefficients of $R(x)$ as small as we please.

Moreover r' can be so taken that $Q(x)$ will have no zero interval. For $Q(x)$ will have no zero interval if $f(x) - R(x)$ has, like $f(x)$, ν zero intervals. But consider the polynomial $f_t(x) = f(x) - tR(x)$, where t is supposed to vary (through rational values) from 0 to 1, and therefore $f_t(x)$ from $f(x)$ to $f(x) - R(x)$. It will be proved in the following section that the number of the zero intervals of a polynomial with variable coefficients changes only when the coefficients pass through values for which the discriminant of the polynomial vanishes (that is, becomes less than any assignable number). Hence, as t varies from 0 to 1, the number of the zero intervals of $f_t(x)$ is always ν unless Disc $f_t(x)$ vanishes. But Disc $f_t(x)$ can be reduced to the form $D + \psi$, where ψ is an integral function of the coefficients of $f(x)$ and $tR(x)$, and therefore cannot vanish if $|\psi| < |D|$. But it will readily be seen that $|\psi| < |D|$ if, g denoting the greatest coefficient in the development of $D[|a_0|, \cdots |a_{n-\nu}|, |a_{n-\nu+1}| + 1/r', \cdots, |a_n| + 1/r']$ in powers of $1/r'$, we take r' such that $1/r' < |D|/(|D| + g)$. Therefore if $r' > (|D| + g)/|D|$, $Q(x)$ will have no zero interval.

6. But it remains to show that for a polynomial with variable coefficients the vanishing of the discriminant is the necessary condition for a change in the number of zero intervals. Kronecker establishes this theorem by aid of the notion of the characteristic of an equation or integral function.

Let $V(x)$ and $W(x)$ be two polynomials of the same degree with rational coefficients and such that $R(V, W)$, $R(V, V')$, and $R(W, W')$ are all different from 0; and let

$$(1) \qquad \Delta(x) = \begin{vmatrix} V(x) & W(x) \\ V'(x) & W'(x) \end{vmatrix}.$$

By the method of § 4 determine an integer s such that no interval of the magnitude $1/s$ contains two zero intervals of $\varphi(x) = V(x)\, W(x)$, and, representing the interval in which all the zero intervals of $\varphi(x)$ lie by $(-c, c)$, divide $(-c, c)$ into equal intervals (x_1, x_2), (x_2, x_3), $\cdots$ of the magnitude $1/s$. Represent each of these intervals which is a zero interval of $V(x)$ by any number ξ belonging to the interval, and let η have the corresponding meaning for $W(x)$ and ζ for $\varphi(x)$. Finally let the symbol $\operatorname{sgn} f(x)$ mean 1 or -1 according as $f(x)$ is positive or negative.

If (x_i, x_{i+1}) be a zero interval of $\varphi(x)$, we have, by § 2, $\operatorname{sgn} \varphi'(\zeta) = \operatorname{sgn}[\varphi(x_{i+1}) - \varphi(x_i)] = \frac{1}{2}[\operatorname{sgn} \varphi(x_{i+1}) - \operatorname{sgn} \varphi(x_i)]$. But if (x_i, x_{i+1}) be not a zero interval, $\operatorname{sgn} \varphi(x_{i+1}) - \operatorname{sgn} \varphi(x_i) = 0$. Therefore if Σ_ζ denote a sum extended over all the zero intervals of $\varphi(x)$, we have, since the degree of $\varphi(x)$ is even,

$$\Sigma_\zeta \operatorname{sgn} \varphi'(\zeta) = \tfrac{1}{2}\operatorname{sgn} \varphi(c) - \tfrac{1}{2}\operatorname{sgn} \varphi(-c) = 0,$$

that is,

$$\Sigma_\xi \operatorname{sgn} V'(\xi)\, W(\xi) + \Sigma_\eta \operatorname{sgn} V(\eta)\, W'(\eta) = 0, \tag{2}$$

the sums Σ_ξ and Σ_η being extended over all the zero intervals of $V(x)$ and $W(x)$ respectively. And from (1) and (2) it at once follows that

$$\Sigma_\xi \operatorname{sgn} \Delta(\xi) = \Sigma_\eta \operatorname{sgn} \Delta(\eta). \tag{3}$$

The integer or fraction $\mathrm{X}(V, W)$ defined by the equations

$$\mathrm{X}(V, W) = -\tfrac{1}{2}\Sigma_\xi \operatorname{sgn} \Delta(\xi) = -\tfrac{1}{2}\Sigma_\eta \operatorname{sgn} \Delta(\eta) \tag{4}$$

is called the *characteristic* of the functions $V(x)$ and $W(x)$. In particular

$$\mathrm{X}(V, V + V') = \tfrac{1}{2}\Sigma_\xi \operatorname{sgn} V'^2(\xi). \tag{5}$$

Therefore since $\operatorname{sgn} V'^2(\xi) = 1$, the number of the zero intervals of any polynomial $V(x)$ which has no multiple factors is $2\mathrm{X}(V, V + V')$, or, more simply, $2\mathrm{X}(V, V')$, if in defining $\mathrm{X}(V, V')$ we restrict ourselves to the first of the equations (4). It is convenient to call $\mathrm{X}(V, V')$ the characteristic of $V(x)$.

If $V(x) = \Sigma_{i=0}^{n} v_i x^i$, $W(x) = \Sigma_{i=0}^{n} w_i x^i$, and the coefficients v_i, w_i are supposed to vary continuously (through rational values), the only possibilities of a change in the value of $\mathrm{X}(V, W)$ are such as present themselves when the v_i, w_i pass through values for which $\Delta(V, W) = V(x)\, W'(x) - V'(x)\, W(x)$ vanishes (that is, becomes numerically less than every assignable

positive number δ) at the same time as either $V(x)$ or $W(x)$ that is, when (1) $V(x) = 0$ and $W(x) = 0$, or (2) $V(x) = 0$ and $V'(x) = 0$, or (3) $W(x) = 0$ and $W'(x) = 0$. But, for all values of the parameter t,

$$\begin{vmatrix} V(x), W(x) \\ V'(x), W'(x) \end{vmatrix} = \begin{vmatrix} V(x), W(x) - tV(x) \\ V'(x), W'(x) - tV'(x) \end{vmatrix} = \begin{vmatrix} V(x) - tW(x), W(x) \\ V'(x) - tW'(x), W'(x) \end{vmatrix}$$

and therefore $\mathrm{X}(V, W) = \mathrm{X}(V, W - tV) = \mathrm{X}(V - tW, W)$, and the only one of the possible conditions for a change of value (1), (2), (3) which is common to these three expressions for $\mathrm{X}(V, W)$ is (1) $V(x) = 0$ and $W(x) = 0$; since for any values of v_i, w_i, and x for which $V(x)\,W'(x) - V'(x)\,W(x)$ vanishes (except those for which both $V(x)$ and $W(x)$ vanish), t may be so chosen that both $W(x) - tV(x)$ and $V(x) - tW(x)$ are different from 0. Therefore

The characteristic $\mathrm{X}(V, W)$ *of two polynomials* $V(x)$, $W(x)$ *with variable coefficients can change only when the resultant* $R(V, W)$ *vanishes.*

The characteristic $\mathrm{X}(V, V')$ *of a polynomial* $V(x)$ *with variable coefficients, and therefore the number of the zero intervals of* $V(x)$, *can change only when the discriminant* $D(V)$ *vanishes.*

7. From the equations (4) and (6) of § 5 it follows that

$$(1) \quad f(x) - S(x)/r' = (x - \xi_1)(x - \xi_2) \cdots (x - \xi_\nu)\, Q(x).$$

We have therefore shown for any polynomial $f(x) = \Sigma_{i=0}^{n} a_i x^{n-i}$ which has rational coefficients and ν zero intervals that by a change in the coefficients $a_{n-\nu+1}, \cdots, a_n$ which is as small as we please the polynomial becomes resolvable into a product of ν linear factors and a factor $Q(x)$ which has no zero intervals, all these factors having rational coefficients. For the case in which $\nu = n$, the proof of Kronecker's theorem is now complete. But for the case in which $\nu < n$, it remains to prove that, by another variation of the coefficients as small as we please, $Q(x)$ becomes resolvable into a product of quadratic factors with rational coefficients.

8. Let $Q(x) = a_0 F(x)$ and $n - \nu = m$. Then $F(x)$ may be written in the form

$$(1) \qquad F(x) = x^m - c_1 x^{m-1} + \cdots + (-1)^m c_m.$$

This function has rational coefficients, no zero intervals, and its discriminant D is not 0.

Form the function

$$F_1(x) = (x - x_1)(x - x_2) \cdots (x - x_m)$$
$$= x^m - f_1 x^{m-1} + \cdots + (-1)^m f_m, \tag{2}$$

where $x_1, x_2, \cdots, x_m$ are independent variables.

Select any two of these variables, say x_1 and x_2, and form the function

$$g = p(x_1 + x_2) + x_1 x_2, \tag{3}$$

where p represents a rational integer.

Of the $m!$ permutations of $x_1, x_2, \cdots, x_m$ there are $2(m-2)!$ which leave g interchanged. But if we apply all the $m!$ permutations we obtain a system of $\rho = m(m-1)/2$ conjugate functions

$$g_1, g_2, \cdots, g_\rho \tag{4}$$

of which one, say g_1, is g itself. Moreover since the difference of any two of these functions, namely

$$p(x_h + x_k) + x_h x_k - p(x_i + x_j) - x_i x_j,$$

cannot vanish for all values of p unless $x_h = x_i$, $x_k = x_j$ or $x_h = x_j$, $x_k = x_i$, we can always so determine p (and we shall suppose this to have been done) that the functions $g_1, g_2, \cdots, g_\rho$ are all different, provided, of course, that the discriminant of $F_1(x)$ is not 0. The function $g = g_1$ is therefore a root of an equation of the ρth degree

$$\prod_{i=1}^{\rho} (z - g_i) = \phi(z, f_1, f_2, \cdots, f_m) = 0 \tag{5}$$

with coefficients which are rational integral functions of the f_i's. The discriminant of this equation is not 0 and the equation is irreducible in the domain of rationality of the f_i's.

Every rational integral symmetric function of the variables x_1, x_2, is a rational integral function of g, of degree $\rho - 1$ at most, with coefficients which are rational in $f_1, f_2, \cdots, f_m$.

For let γ be such a function and let $\gamma_1, \gamma_2, \cdots, \gamma_\rho$, where $\gamma_1 = \gamma$, be its conjugate functions arranged to correspond to the conjugate functions $g_1, g_2, \cdots, g_\rho$. By Lagrange's interpolation formula, the function $f(z)$ whose values for $z = g_1, g_2, \cdots, g_\rho$ are $\gamma_1, \gamma_2, \cdots, \gamma_\rho$ respectively is

$$f(z) = \sum_{i=1}^{\rho} \frac{\phi(z)}{z - g_i} \frac{\gamma_i}{\phi'(g_i)} = \frac{s_0 z^{\rho-1} + s_1 z^{\rho-2} + \cdots + s_{\rho-1}}{\phi'(g_1)\phi'(g_2) \cdots \phi'(g_\rho)}, \tag{6}$$

where the coefficients $s_0, s_1, \cdots, s_{\rho-1}$ are rational integral functions of $f_1, \cdots, f_m$ and the denominator is the discriminant of $\phi(z) = 0$. Therefore

$$\gamma = f(g) = \frac{s_0 g^{\rho-1} + s_1 g^{\rho-2} + \cdots + s_{\rho-1}}{\text{Disc } \phi(z)}. \tag{7}$$

The function $F_1(x)$ can be reduced to the form

$$F_1(x) = [x^2 - (x_1 + x_2)x + x_1 x_2]Q(x, x_1, x_2, f_1, \cdots, f_m),$$

where Q is symmetric with respect to x_1, x_2, and therefore, by the theorem just demonstrated, to the form

$$F_1(x) = [x^2 - \varphi(g)x + \psi(g)]Q(x, g, f_1, \cdots, f_m), \tag{8}$$

where $\varphi(g)$, $\psi(g)$ and Q are rational with respect to the f_i's and g and integral with respect to g. But from (8) the congruence $F_1(x) - [x^2 - \varphi(z)x + \psi(z)]Q[x, z, f_1, \cdots, f_m] \equiv 0 \pmod{z - g}$ follows, and from this in turn, since $\phi(z, f_1, \cdots, f_m)$ is the irreducible function in the domain of rationality of the f_i's of which $z - g$ is a factor, the congruence

$$F_1(x) - [x^2 - \varphi(z)x + \psi(z)]Q(x, z, f_1, \cdots, f_m) \equiv 0 \pmod{\phi(z, f_1, \cdots, f_m)}. \tag{9}$$

If in (9), we substitute for the f_i's the corrresponding coefficients of the given function $F(x)$, we obtain

$$F(x) - [x^2 - \varphi(z)x + \psi(z)]Q[x, z, c_1, \cdots, c_m] \equiv 0 \pmod{\phi(z, c_1, \cdots, c_m)}, \tag{10}$$

where the coefficients of $\varphi(z)$ and $\psi(z)$ are now rational functions of the c_i's. But from (10) it follows that if we carry out the complete division of $F(x)$ by $x^2 - \varphi(z)x + \psi(z)$ we shall obtain a remainder which can be reduced to the form $[r_0(z)\, x + r_1(z)]\phi(z)$, where $r_0(z)$, $r_1(z)$ are integral functions of z, and $\phi(z)$ stands for $\phi(z, c_1, \cdots, c_m)$. Hence the result of this division may be written

$$F(x) - [x^2 - \varphi(z)x + \psi(z)]Q_1(x, z, c_1, \cdots, c_m) = [r_0(z)x + r_1(z)]\phi(z). \tag{11}$$

Since by hypothesis Disc $F(x)$ is different from 0, so also is Disc $\phi(z)$. Therefore $\phi(z)$ has no multiple factors. We proceed to show that $\phi(z)$ has $\mu = m/2$ zero intervals.

With the given function $F(x)$, which has no zero intervals, associate the function $\overline{F}(x) = F(x) + \tau F'(x)$, where τ is a constant so chosen that Disc $[F(x) + tF'(x)]$ does not vanish as t varies from 0 to τ, and therefore $\overline{F}(x)$, like $F(x)$, has no zero intervals. Then $\mathrm{X}(F, F) = 0$ (§ 6).

Again let $f(x) = \Pi[x^2 - 2a_i x + (a_i^2 + b_i^2)]$, where the a_i, b_i are any pairs of rational integers, all different; also let $\overline{f}(x) = f(x) + \tau f'(x)$ be a polynomial related to $f(x)$ as is $\overline{F}(x)$ to $F(x)$. Then $\mathrm{X}(f, \overline{f}) = 0$.

Finally form the pair of functions

$$(12) \qquad \begin{aligned} F_t(x) &= (1-t)\,f(x) + t\,F(x), \\ \overline{F}_t(x) &= (1-t)\,\overline{f}(x) + t\,\overline{F}(x), \end{aligned} \qquad (0 \leqq t \leqq 1),$$

Evidently $F_t(x)$ and $\overline{F}_t(x)$, as thus defined, have positive values only. Therefore, if they have a common factor for any value of t in the interval (0, 1), this factor must be of even degree, hence of the second degree at least. But if $F_t(x)$ and $\overline{F}_t(x)$ are to have a common factor of the second degree, t must satisfy not only the condition $R(F_t, \overline{F}_t) = 0$ but also the condition $\partial R(F_t, \overline{F}_t)/\partial a = 0$, where a denotes the term in either $F_t(x)$ or $\overline{F}_t(x)$ which does not involve x;* and therefore the coefficients of $F(x)$ and $f(x)$ must satisfy the condition $R[R(F_t, \overline{F}), \partial R(\overline{F}_t, F_t)/\partial a] = 0$. This last condition is not an identity. Hence the numbers a_i, b_i on which the coefficients

* It may be proved as follows that if the functions $V(x) = v_0 + v_1 x + \cdots + v_m x^m$ and $W(x) = w_0 + w_1 x + \cdots + w_n x^n$ are to have a common factor of the second degree, to the condition $R = R(V, W) = 0$ there must be added the condition $\partial R/\partial v_0 \equiv 0$. For the functions $V(x) - V(u)$ and $W(x) - W(u)$ have the common factor $x - u$, and therefore

$$R(v_0 - V(u), v_1, \cdots, v_m, w_0 - W(u), w_1, \cdots, w_n) = 0$$

identically. But if we take the derivative of this identity with respect to v_1, develop the result in powers of $V(u)$ and $W(u)$, and then replace u by x, we obtain an identity which is equivalent to the congruence

$$(1) \qquad \frac{\partial R}{\partial v_0} x + \frac{\partial R}{\partial v_1} \equiv 0 \qquad (\text{modd. } V(x), W(x)).$$

Hence if $V(x)$ and $W(x)$ have a common factor of the second degree both $\partial R/\partial v_0 = 0$ and $\partial R/\partial v_1 = 0$. But $\partial R/\partial v_1 = 0$ is not an independent condition; for since $R = 0$ is a sufficient as well as a necessary condition for the existence of a common factor of at least the first degree, if both $R = 0$ and $\partial R/\partial v_0 = 0$ it follows from (1) that $\partial R/\partial v_1 = 0$.

of $f(x)$ depend may be so chosen that the condition is not satisfied. Suppose such a choice to have been made. We then have $R(F_t, \bar{F}_t) \neq 0$, and therefore Disc $F_t(x) \neq 0$, for all values of t in the interval (0, 1).

Let $\phi_0(z)$ be the polynomial obtained from $\phi(z, f_i)$ by replacing each $(-1)^i f_i$ by the corresponding coefficient of $f(x)$ expressed as a polynomial in x. The equation $\phi_0(z) = 0$ has μ real roots, namely, $2pa_i + (a_i^2 + b_i^2)$ $(i = 1, 2, \cdots, \mu)$. Hence $X(\phi, \phi') = \mu/2$.

As t varies from 0 to 1, and therefore $F_t(x)$ from $f(x)$ to $F(x)$, the function

$$\phi_t(z) = (1 - t)\phi_0(z) + t\phi(z) \tag{13}$$

varies from $\phi_0(z)$ to $\phi(z)$. Since Disc $F_t(x)$ never vanishes during this variation, the like is true of Disc $\phi_t(z)$. Therefore since $X(\phi_t, \phi_t')$ is $\mu/2$ when $t = 0$ it is also $\mu/2$ when $t = 1$. In other words, $\phi(z)$ has $\mu = m/2$ zero intervals, as was to be proved.

Therefore, if s denote an integer (determined as in § 4) such that no z-interval of the magnitude $1/s$ contains zero intervals of both $\phi(z)$ and $\phi'(z)$, and if $D = \text{Disc } \phi(z)$, we can, as in § 5, find μ intervals

$$(\zeta_1', \zeta_1''), (\zeta_2', \zeta_2''), \cdots, (\zeta_\mu', \zeta_\mu'') \tag{14}$$

of the magnitude $1/rs \mid D \mid$ such that

$$\phi(\zeta_i')\phi(\zeta_i'') < 0 \text{ and } \mid \phi(z) \mid < 1/r \text{ for } \zeta_i' \leqq z \leqq \zeta_i'' \quad (i = 1, 2, \cdots, \mu),$$

where r is an integer which may be taken as great as we please.

If ζ_i denote any number in (ζ_i', ζ_i'') and we set $z = \zeta_i$ in (11), we have

$$\begin{aligned} F(x) - [x^2 - \varphi(\zeta_i)x + \psi(\zeta_i)]Q(x, \zeta_i, c_1, \cdots, c_m) \\ = [r_0(\zeta_i)x + r_1(\zeta_i)]\phi(\zeta_i), \end{aligned} \tag{15}$$

where the right member is a linear function of x with rational coefficients which are numerically less than $r_0(\zeta_i)/r$ and $r_1(\zeta_i)/r$ respectively and can therefore be made as small as we please by a suitable choice of r. Moreover, as in § 5, we can determine an integer $\bar{r}$ such that for $r > \bar{r}$ each of the products $[x^2 - \varphi(\zeta_i)x + \psi(\zeta_i)]Q(x, \xi_i, c_1, \cdots, c_m)$ $(i = 1, 2, \cdots, \mu)$ is, like $F(x)$, without zero intervals.

It has therefore been shown that we can find μ quadratic factors with rational coefficients and without zero intervals

$$G_1 = x^2 - \phi(\zeta_1)x + \psi(\zeta_1),\ G_2 = x^2 - \psi(\zeta_2)x + \psi(\zeta_2),\ \cdots,$$
$$G_\mu = x^2 - \psi(\zeta_\mu)x + \psi(\zeta_\mu)$$

such that each of the remainders Rem (F/G_i) will be as small as we please. But from this it follows that the remainder Rem $(F/G_1, G_2, \cdots, G_\mu)$ can be made as small as we please.

For if no two of the functions G_i have a common factor, and we shall show that this is the case, we can find two integral functions P_k, Q_k, such that

$$P_k \cdot G_1G_2 \cdots G_{k-1}G_{k+1} \cdots G_\mu + Q_kG_k = R_k, \tag{16}$$

where P_k is of the first degree at most, both P_k and Q_k are integral with respect to the coefficients of the functions G_i, and R_k is the resultant $R[G_k, G_1G_2 \cdots G_{k-1}G_{k+1} \cdots G_\mu]$.

Let $P_k' = P_k/R_k$, so that P_k' has fractional coefficients, but such as involve only factors of the form $R(G_k, G_1)$, $R(G_k, G_2)$, $\cdots$ in their denominators. It will then follow from (16) that

$$P_k' \cdot G_1G_2 \cdots G_{k-1}G_{k+1} \cdots G_\mu \equiv 1 \pmod{G_k}$$

and, therefore, if $G = G_1G_2 \cdots G_\mu$, that

$$\sum_{k=1}^{\mu} P_k' \cdot G_1G_2 \cdots G_{k-1}G_{k+1} \cdots G_\mu \equiv 1 \pmod{G}.$$

But each term in the first member of this congruence is of the degree $m - 1$ at most, while G is of degree m. Hence

$$\sum_{k=1}^{\mu} P_k' \cdot G_1G_2 \cdots G_{k-1}G_{k+1} \cdots G_\mu = 1,$$

from which it follows, by multiplying throughout by F/G, that

$$\frac{F}{G} = \frac{FP_1'}{G_1} + \frac{FP_2'}{G_2} + \cdots + \frac{FP_\mu'}{G_\mu}, \tag{17}$$

and therefore that

$$\operatorname{Rem}\left(\frac{F}{G}\right) = G_2G_3 \cdots G_\mu \operatorname{Rem}\left(\frac{FP_1'}{G_1}\right) + \cdots + G_1G_2 \cdots G_{\mu-1} \operatorname{Rem}\left(\frac{FP_\mu'}{G_\mu}\right). \tag{18}$$

Rem (F/G_k) is $[r_0(\zeta_k)x + r_1(\zeta_k)]\phi(\zeta_k)$. Hence Rem (FP_k'/G_k) is of the form $(a_kx + b_k)\phi(\zeta_k)/R_k$ where a_k and b_k are integral functions of the ζ_i's. Therefore since each R_k is different from 0 and each $|\phi(\zeta_k)|$ is less than $1/r$, Rem (F/G) is a polynomial in x of degree $m-1$ at most with coefficients which by a suitable choice of $r(>\bar{r})$ and the consequent choice of the numbers ζ_k can be made as small as we please.

9. It remains to prove that the resultant of no two of the functions G_i, say the functions $G_1 = x^2 - \phi(\zeta_1)x + \psi(\zeta_1)$ and $G_2 = x^2 - \varphi(\zeta_2)x + \psi(\zeta_2)$ can be 0, that is, less numerically than every assignable number δ. This resultant may be reduced to the form

$$\begin{aligned} 16\,R(G_1, G_2) &= (\varphi(\zeta_1) - \varphi(\zeta_2))^4 + 2(\varphi(\zeta_1) - \varphi(\zeta_2))^2 \\ &\quad \times [4\psi(\zeta_1) - \varphi^2(\zeta_1) + 4\psi(\zeta_2) - \varphi^2(\zeta_2)] \\ &\quad + [4(\psi(\zeta_1) - \psi(\zeta_2)) - (\varphi^2(\zeta_1) - \varphi^2(\zeta_2)]^2. \end{aligned} \tag{19}$$

Both $4\psi(\zeta_1) - \varphi^2(\zeta_1)$ and $4\psi(\zeta_2) - \varphi^2(\zeta_2)$ are positive. Hence $R(G_1, G_2)$ cannot be numerically less than every assignable number unless the same is true of both $\varphi(\zeta_1) - \varphi(\zeta_2)$ and $\psi(\zeta_1) - \psi(\zeta_2)$. But, as we proceed to show, at least one of these differences is numerically greater than a definite positive number.

In defining the function $g = p(x_1 + x_2) + x_1x_2$ we subjected the integer p to the single condition implied in the requirement that the discriminant of the equation $\phi(z, f_i) = 0$ of which g is a root be not 0. We shall now define p more closely, as follows. Let

$$p = 2q + t \quad \text{where} \quad t \geqq q \geqq 1, \tag{20}$$

determining the integer q also so that the equation $\phi_1(z, f_i) = 0$ of which the function $g_1 = q(x_1 + x_2) + x_1x_2$ is a root shall not have a zero discriminant. This function g_1 is a rational integral function of g; let $g_1 = \theta(g, f_i)$. The equations $\phi(z, f_i) = 0$ and $\phi_1[\theta(z), f_i] = 0$, of which the first is irreducible, have the root g in common. Therefore $\phi_1[\theta(z), f_i]$ is exactly divisible by $\phi(z, f_i)$. Hence, if in these functions we replace the f_i's by the corresponding coefficients c_i of $F(x)$ and call the resulting polynomials $\phi(z)$ and $\phi_1[\theta(z)]$, we have

$$\phi_1[\theta(z)] = \phi(z)\,Q(z), \tag{21}$$

where $Q(z)$ is integral and all the coefficients are rational in the coefficients of $F(x)$. If, therefore, any interval (z', z'') con-

tains a zero interval of $\phi(z)$, the interval to which the values which $\theta(z)$ takes in (z', z'') belong will contain a zero interval of $\phi_1(z)$.

Let $1/s_1$ represent the magnitude of an interval which cannot contain zero intervals of both $\phi_1(z)$ and $\phi_1'(z)$, the integer s_1 being determined as in § 5; let u be an integer greater than all the values of $|[\theta(z+\sigma)-\theta(z)]/\sigma|$, $|\sigma| \leqq 1$, in the interval $(-c_1, c_1)$ in which all the zero intervals of $\phi_1(z)$ lie; finally, to the conditions already stated for determining s for $\phi(z)$ add $s \geqq s_1 u$. We shall then have for all values of z in $(-c_1, c_1)$

$$(22) \qquad |\theta(z+\sigma)-\theta(z)| < |\sigma|\frac{s}{s_1} \quad (|\sigma| \leqq 1).$$

Therefore to values of z belonging to intervals of magnitude $1/s$ and $1/rs|D|$ respectively there correspond values of $\theta(z)$ belonging to intervals of magnitude $1/s_1$ and $1/rs_1|D|$. The numbers ζ_1 and ζ_2 belong to zero intervals (ζ_1', ζ_1'') and (ζ_2', ζ_2'') of $\phi(z)$ of magnitude $1/rs|D|$. Hence $\theta(\zeta_1)$ and $\theta(\zeta_2)$ belong to intervals of magnitude $1/rs_1|D|$ which it was shown above are zero intervals of $\phi_1(z)$. But from this and the fact that no interval of the magnitude $1/s_1$ can contain zero intervals of both $\phi_1(z)$ and $\phi_1'(z)$ it follows (see § 5) that $|\theta(\zeta_1)-\theta(\zeta_2)| > 1/s_1$. Therefore, if for brevity we set $\theta(\zeta_1)=a_1$, $\theta(\zeta_2)=a_2$, $\zeta_1=b_1$, $\zeta_2=b_2$, we have

$$(23) \qquad |a_1-a_2| > \frac{1}{s_1} \quad \text{and} \quad |b_1-b_2| > \frac{1}{s}.$$

Since

$$p(x_1+x_2)+x_1x_2=g, \quad q(x_1+x_2)+x_1x_2=\theta(g),$$

$$x_1+x_2=\varphi(g), \quad x_1x_2=\psi(g)$$

we have, on replacing g by ζ_i,

$$p\varphi(\zeta_i)+\psi(\zeta_i)=\zeta_i, \quad q\varphi(\zeta_i)+\psi(\zeta_i)=\theta(\zeta_i)$$

and therefore

$$(24) \qquad \varphi(\zeta_i)=\frac{\theta(\zeta_i)-\zeta_i}{q-p}, \quad \psi(\zeta_i)=\frac{p\theta(\zeta_i)-q\zeta_i}{p-q}.$$

It therefore only remains to prove that one or other of the quantities

$$(25) \quad d_1=a_1-a_2-(b_1-b_2), \quad d_2=p(a_1-a_2)-q(b_1-b_2)$$

must be numerically greater than some definite positive number. We shall show that $1/s_1$ is such a number.

For if $|d_1| < 1/s_1$ then $b_1 - b_2 = a_1 - a_2 \pm \delta/s_1$ where $0 \leqq \delta < 1$, and therefore $d_2 = (p-q)(a_1 - a_2) \pm \delta q/s_1$, which by (20) and (23) is numerically greater than $1/s_1$.

And if $|d_2| < 1/s_1$, then $q(b_1 - b_2) = p(a_1 - a_2) \pm \delta'/s_1$, where $0 \leqq \delta' < 1$ and therefore $d_1 = -(q+t)(a_1 - a_2)/q \pm \delta'/qs_1$, which again is numerically greater than $1/s_1$.

10. Kronecker's theorem has now been demonstrated. For referring to equations (1) of § 7 and (18) of § 8 it will be seen that if we set

$$F_{r'r}(x) = S(x)/r' + a_0(x-\xi_1)(x-\xi_2)\cdots(x-\xi_\nu)\cdot \operatorname{Rem} F/G$$

we shall have

$$\begin{aligned} f(x) - F_{r'r}(x) = a_0(x-\xi_1)(x-\xi_2)\cdots(x-\xi_\nu) \\ \times (x^2 - \varphi(\zeta_1)x + \psi(\zeta_1))\cdots(x^2 - \varphi(\zeta_\mu)x + \psi(\zeta_\nu)). \end{aligned} \tag{1}$$

We have given methods by which ν, the numbers ξ_i and ζ_i, the functions $\varphi(\zeta)$, $\psi(\zeta)$, and the remainder function $F_{r'r}(x)$ may be determined from the coefficients of the given polynomial $f(x) = \Sigma_{i=0}^{n} a^i x^{n-i}$ by a finite number of rational operations. The function $F_{r'r}(x)$ is of degree $n-1$ at most. Its coefficients are all numerically less than k/r' or k/r, where k is a determinable positive number, and can therefore be made as small as we please by assigning sufficiently great values to r' and r and then making the corresponding determinations of the numbers ξ_i and ζ_i. And when the coefficients of $f(x)$ are subjected to the variation which consists in subtracting from them the corresponding coefficients of $F_{r'r}(x)$, the resulting polynomial is a product of ν linear and $(n-\nu)/2$ quadratic factors, all with rational coefficients.

In conclusion it may be remarked that if recognition be accorded the irrational numbers, and we represent the limits which the numbers ξ_i and ζ_i approach as r' and r approach infinity by ξ_i' and ζ_i' respectively, it will follow from (1), since $\lim F_{r'r}(x) = 0$, that

$$\begin{aligned} f(x) = a_0(x-\xi_1')\cdots(x-\xi_\nu')(x^2 - \varphi(\zeta_1')x + \psi(\zeta_1')) \\ \cdots (x^2 - \varphi(\zeta_\mu')x + \psi(\zeta_\mu')) \end{aligned} \tag{2}$$

so that from the result of Kronecker's discussion there follows, for numerical equations, a proof of the fundamental theorem of algebra as ordinarily stated.

PRINCETON UNIVERSITY,
December, 1913.

TWO CONVERGENCY PROOFS.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society, December 30, 1913.)

1. *Introduction.*

IN the study of automorphic functions defined within a fundamental domain G formed by two non-intersecting circles in the "elliptic" case and by two tangent circles in the "trigonometric" case, it is necessary to prove the convergence of certain fundamental series, as has been shown by Schottky recently.* In the first case the substitutions of the cyclic group associated with it may be written in the form

$$\frac{x_\lambda - a}{x_\lambda - b} = q^\lambda \frac{x - a}{x - b}, \tag{1}$$

where λ may assume all integral real values between $-\infty$ and $+\infty$ and where a and b are invariant in all substitutions of the group. When x describes the circle K_{-1} of the fundamental region G, x_1 describes the circle K_1 forming the other boundary of G. x_λ describes a circle K_λ and all circles K_λ of the pencil with a and b as limiting points divide the whole x-plane into a set of regions corresponding to the substitutions of the group.† Schottky bases his proof of the convergence of the series Σr_λ of the radii of the circles K_λ on the invariance of the expression

$$\frac{(r_\lambda^2 + r_{\lambda+1}^2 - e_\lambda^2)^2}{4r_\lambda^2 r_{\lambda+1}^2} - 1.$$

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* "Ueber die Funktionenklasse, die der Gleichung $F\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) = F(x)$ genügt," *Journal für reine und angewandte Mathematik*, vol. 143, pp. 1–24 (May, 1913).

† See also Klein-Fricke, Vorlesungen über die Theorie der elliptischen Modulfunctionen, vol. 1, pp. 163–207.

This is a positive quantity and is equal to $-\sin^2\phi$, where ϕ is the imaginary angle of intersection of K_{-1} and K_1 and appears also in the invariant distance

$$|a-b| = \frac{2r_\lambda r_{\lambda+1}}{e_\lambda}\sqrt{\frac{(r_\lambda^2 + r_{\lambda+1}^2 - e_\lambda^2)^2}{4r_\lambda^2 r_{\lambda+1}^2}} - 1.$$

In the second case the substitutions may be written in the form

$$\frac{A}{x_\lambda - a} = \frac{A}{x-a} + \lambda. \tag{2}$$

The domains corresponding to the substitutions of the group are bounded by a parabolic pencil of circles with the common point of tangency a. Taking for X and X' any two points of the x-plane different from a, Schottky proves the absolute convergence of the series

$$\sum_{\lambda=-\infty}^{+\infty}(X_\lambda - X_\lambda') \tag{3}$$

by first proving that of $\Sigma(x_\lambda - x_\lambda')$, in which x is any point different from a, and x' a point between x and x_1 on the circle through a, x and x_1. For both the series Σr_λ and (3) I shall give direct proofs.

2. *Convergence of* Σr_λ.

In the substitution

$$\frac{x_\lambda - a}{x_\lambda - b} = q^\lambda \frac{x-a}{x-b} \qquad (|q| \neq 1),$$

assume first $|q| = \rho > 1$ and let x describe the circle K_{-1} around a, so that

$$\left|\frac{x-a}{x-b}\right| = \sigma < 1.$$

Assuming furthermore $\rho\sigma > 1$, then x_1 describes the circle K_1 about b, so that the fundamental domain G bounded by K_{-1} and K_1 extends to infinity. All circles K_λ with $\lambda = 1, 2, 3, \cdots$ enclose b; all circles K_λ with $\lambda = 0, -1, -2, -3, \cdots$ enclose a. Designating the distance of the centers of K_{-1} and K_1 by e, the radius of K_λ for positive values of λ is easily found from

$$\left|\frac{x_\lambda - a}{x_\lambda - b}\right| = \sigma\rho^\lambda$$

as

$$r_\lambda = \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1}, \tag{4}$$

and in case $\lambda \leqq 0$ as

$$r_\lambda = \frac{e\sigma\rho^\lambda}{1 - \sigma^2\rho^{2\lambda}}. \tag{5}$$

To prove first the convergence of

$$\sum_{\lambda=1}^{\infty} \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1}, \tag{6}$$

we write

$$r_\lambda = \frac{e}{\sigma}\frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^2}.$$

Now, since $\sigma < 1$, $1/\sigma^{2\lambda} > 1/\sigma^2$, and from $\sigma\rho > 1$, $\rho > 1/\sigma$, we have

$$\rho^{2\lambda} - \frac{1}{\sigma^{2\lambda}} > 0$$

and

$$\rho^{2\lambda} - \frac{1}{\sigma^{2\lambda}} < \rho^{2\lambda} - \frac{1}{\sigma^2}.$$

Consequently

$$\frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^2} < \frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^{2\lambda}}.$$

But

$$\frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^{2\lambda}} = \frac{\rho^\lambda}{(\rho^2 - 1/\sigma^2)(\rho^{2(\lambda-1)} + \rho^{2(\lambda-2)} \cdot 1/\sigma^2 + \cdots)}$$

$$< \frac{\rho^\lambda}{(\rho^2 - 1\sigma^2)/\rho^{2\lambda-1}},$$

so that finally

$$\sum_{\lambda=1}^{\infty} \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1} < \frac{e}{\sigma(\rho^2 - 1/\sigma^2)} \sum_{\lambda=1}^{\infty} \frac{1}{\rho^{\lambda-2}},$$

or

$$\sum_{\lambda=1}^{\infty} \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1} < \frac{e\rho^3}{\sigma(\rho - 1)(\rho^2 - 1/\sigma^2)},$$

i. e., less than a finite positive quantity. For negative values of λ, $\lambda = -\mu$ $(\mu \geqq 0)$, we have the series

$$\sum_{\mu=0}^{\infty} r_{-\mu} = \sum_{\mu=0}^{\infty} \frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - \sigma^2}.$$

As $\sigma^2 < 1$,

$$\frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - \sigma^2} < \frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - 1}.$$

But

$$\rho^{2\mu} - 1 = (\rho^2 - 1)(\rho^{2(\lambda-1)} + \rho^{2(\lambda-2)} + \cdots) > (\rho^2 - 1)\rho^{2(\lambda-1)},$$

so that

$$\sum_{\mu=1}^{\infty} \frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - \sigma^2} < \sum_{\mu=0}^{\infty} \frac{e\sigma\rho^{\mu}}{(\rho^2 - 1)\rho^{2(\lambda-1)}},$$

or finally

$$\sum_{\mu=0}^{\infty} r_{\mu} < \frac{e\sigma\rho^2}{(\rho - 1)(\rho^2 - 1)},$$

less than a positive finite quantity.

Consequently

$$\sum_{\lambda=-\infty}^{+\infty} r_{\lambda} = \sum_{\lambda=1}^{+\infty} r_{\lambda} + \sum_{\mu=0}^{+\infty} r_{\mu}$$

is convergent. In a similar manner the convergence can be proved for the sum of the radii of any other loxodromic cyclic group.*

3. *Convergence of $\Sigma(x_\lambda - x_\lambda')$ for a Parabolic Cyclic Group.*

In

$$\frac{A}{x_\lambda - a} = \frac{A}{x - a} + \lambda, \quad \text{put} \quad \frac{A}{x - a} + c = z,$$

where c is any constant. Putting $z - c = \zeta$, then by means of this transformation the x-plane is transformed into the ζ-plane, and we find

$$\zeta_\lambda = \zeta + \lambda$$

* Schottky proved the convergence of the sum of radii for a more general set of circles in an article: "Ueber eine spezielle Function, welche bei einer bestimmten linearen Transformation ihres Argumentes unverändert bleibt," *Journal für reine und angewandte Mathematik*, vol. 101, pp. 231–236 (1887).

and

$$x_\lambda - a = \frac{A}{\zeta + \lambda}, \quad x_\lambda' - a = \frac{A}{\zeta' + \lambda}.$$

Consequently, for $\zeta \neq \zeta'$ and not equal to real integers,

$$(7) \quad \sum_{\lambda=-\infty}^{+\infty} (x_\lambda - x_\lambda') = A(\zeta' - \zeta) \sum_{\lambda=-\infty}^{+\infty} \frac{1}{\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'}.$$

The absolute value of $[\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta']$ has the form $+\sqrt{\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta}$, where α, β, γ, δ are real numbers. We can write

$$+\sqrt{\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta}$$

$$= +\lambda^2 \sqrt{1 + \frac{\alpha}{\lambda} + \frac{\beta}{\lambda^2} + \frac{\gamma}{\lambda^3} + \frac{\delta}{\lambda^4}} = +\lambda^2 \sqrt{1 + \phi(\lambda)}.$$

Assuming an arbitrarily small positive quantity ϵ, it is always possible to choose a positive number n large enough so that for all values of $|\lambda| > n$, $|\phi(\lambda)| < \epsilon$. If, for such values of $\lambda > n$, $\phi(\lambda) > 0$, then

$$+\sqrt{\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta} > \lambda^2$$

and

$$\sum_{\lambda>n}^{+\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|} < \sum_{\lambda>n}^{+\infty} \frac{1}{\lambda^2},$$

for all values of ζ and ζ' different from a real integer. If, for the same values of λ, $\phi(\lambda) < 0$, then

$$\lambda^2 \sqrt{1 + \phi(\lambda)} > \lambda^2 \sqrt{1 - \epsilon},$$

and

$$\sum_{\lambda>n}^{+\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|} < \frac{1}{\sqrt{1 - \epsilon}} \sum_{\lambda>n}^{+\infty} \frac{1}{\lambda^2}.$$

For negative values of λ we have evidently in the first case $\phi(\lambda) < 0$, in the second case $\phi(\lambda) > 0$, so that the two cases for negative values of λ lead again to convergent series. Hence it is always possible to determine a number n large enough so that for all values $\lambda > n$, and all values ζ, ζ' different from an integer, the series

$$\sum_{\lambda}^{+\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|} + \sum_{-\lambda}^{-\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|}$$

is convergent. Consequently in the whole ζ-plane with the exception of the real integers ($\zeta' \neq$ integer), or in the whole x-plane with the exception of $x = a$, and $x' \neq a$,

$$\sum_{\lambda=-\infty}^{+\infty} (x_\lambda - x_\lambda')$$

is a uniformly convergent series of x.

4. *Example.*

If we take in (7) $\zeta' = -\zeta$, we get the function in the ζ-plane

$$\eta = 2A\left(\frac{1}{\zeta} + \sum_{\lambda=1}^{+\infty} \frac{2\zeta}{\zeta^2 - \lambda^2}\right), \tag{8}$$

or, as is well known,

$$\eta = 2A\pi \cot \pi\zeta. \tag{9}$$

Transforming back to the x-plane, i. e., putting

$$\zeta = \frac{A}{x-a}, \quad \eta = \frac{A}{y-a},$$

we find

$$y = a + \frac{1}{2\pi} \tan \frac{\pi A}{x-a}. \tag{10}$$

This function is automorphic. If this is true, it must assume the same value when we replace x by x_λ, or $x - a$ by $x_\lambda - a$. Now

$$\frac{A}{x_\lambda - a} = \frac{A}{x-a} + \lambda$$

and

$$\frac{\pi A}{x_\lambda - a} = \frac{\pi A}{x-a} + \lambda\pi.$$

From this we see immediately that (10) does not change its value under any substitution of the group.

In the z-plane defined by

$$\frac{A}{x-a} + c = z \tag{11}$$

we choose as the fundamental domain the strip between the axis of imaginaries and the line parallel to it at a distance equal

to unity. Taking c within this domain, let z describe the line $z = c + it$, where t is a real variable between $+\infty$ and $-\infty$. To the lines enclosing the strip correspond in the x-plane the two circles K_{-1} and K_1 enclosing the fundamental domain G in the x-plane. As z describes the line $z = c + it$, x describes the common tangent to K_{-1} and K_1 at a, and according as t approaches $+\infty$ or $-\infty$, x approaches, within G, the point a from opposite sides. To find the values of y in (10) as $t = \pm\infty$, we substitute in (10) from (11)

$$\frac{A}{x-a} = z - c = it,$$

so that for points of the common tangent of K_{-1} and K_1

$$y = a + \frac{1}{2\pi}\tan \pi i t. \tag{12}$$

But

$$\tan\phi = \frac{1}{i}\frac{e^{i\phi} - e^{-i\phi}}{e^{i\phi} + e^{-i\phi}},$$

so that

$$\lim_{t=\pm\infty}\{\tan \pi i t\} = \lim_{t=\pm\infty}\left\{\frac{1}{i}\frac{e^{-\pi t} - e^{+\pi t}}{e^{-\pi t} + e^{+\pi t}}\right\} = \mp\frac{1}{i} = \pm i.$$

Thus, as x approaches a within G from different sides, y assumes at a the values $a + \frac{i}{2\pi}$ and $a - \frac{i}{2\pi}$.

UNIVERSITY OF ILLINOIS.

SOME PROPERTIES OF THE GROUP OF ISOMORPHISMS OF AN ABELIAN GROUP.

BY PROFESSOR G. A. MILLER.

(Read before the San Francisco Section of the American Mathematical Society, October 25, 1913.)

As the group of isomorphisms of any abelian group is the direct product of the groups of isomorphisms of its Sylow subgroups, we shall assume that the order of the abelian group G under consideration is p^m, p being any prime number. Moreover, we shall confine our attention to a study of prop-

erties of the Sylow subgroups of order $p^{m'}$ in the group of isomorphisms I of G. The number of these Sylow subgroups can easily be determined by means of the characteristic subgroups of G which involve only operators of order p besides identity. Let I' represent one of these Sylow subgroups and assume that the λ invariants of G are composed of the following numbers: λ_1 which are equal to p^{m_1}, λ_2 which are equal to p^{m_2}, $\cdots$, λ_r which are equal to p^{m_r}. Hence $\lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_r m_r = m$; we shall assume that $m_1 > m_2 > \cdots > m_r$.

It is well known that the characteristic subgroups which are generated by operators of order p contained in G are as follows: One C_1 of order p^{λ_1} which is generated by all the operators of order p which are powers of the operators of highest order contained in G, and is known as the fundamental characteristic subgroup of G*; one C_2 of order $p^{\lambda_1+\lambda_2}$, which is composed of identity and the operators of order p which are powers of the operators of order p^{m_2} contained in G, and includes the preceding; etc. In general, there is one and only one such characteristic subgroup C_α of order $p^{\lambda_1+\lambda_2+\cdots+\lambda_\alpha}$, where α has any value from 1 to r, and this subgroup is generated by the operators of order p which are powers of those of order p^{m_α} contained in G, and it includes all those for smaller values of α.

The number of the Sylow subgroups of order $p^{m'}$ contained in I is clearly equal to the number of ways in which we can choose successively from $C_\alpha (\alpha = 1, 2, \cdots, r)$ one subgroup of each of the orders

$$p^{\lambda_1+\lambda_2+\cdots+\lambda_{\alpha-1}+1},\quad p^{\lambda_1+\lambda_2+\cdots+\lambda_{\alpha-1}+2},\ \ldots,\ p^{\lambda_1+\lambda_2+\cdots+\lambda_{\alpha-1}}$$

such that each one of these subgroups includes all those which precede it and the first includes $C_{\alpha-1}$, C_0 being identity and λ_0 being 0. Hence the number of these Sylow subgroups contained in I is expressed by the following continued product:

$$\frac{p^{\lambda_1}-1}{p-1}\cdot\frac{p^{\lambda_1}-p}{p^2-p}\cdots\frac{p^{\lambda_1}-p^{\lambda_1-1}}{p^{\lambda_1}-p^{\lambda_1-1}}\cdot\frac{p^{\lambda_1+\lambda_2}-p^{\lambda_1}}{p^{\lambda_1+1}-p^{\lambda_1}}\cdot\frac{p^{\lambda_1+\lambda_2}-p^{\lambda_1+1}}{p^{\lambda_1+2}-p^{\lambda_1+1}}\cdots$$
$$\frac{p^{\lambda_1+\lambda_2}-p^{\lambda_1+\lambda_2-1}}{p^{\lambda_1+\lambda_2}-p^{\lambda_1+\lambda_2-1}}\cdots\frac{p^{\lambda}-p^{\lambda-\lambda_r}}{p^{\lambda-\lambda_r+1}-p^{\lambda-\lambda_r}}$$
$$\frac{p^{\lambda}-p^{\lambda-\lambda_r+1}}{p^{\lambda-\lambda_r+2}-p^{\lambda-\lambda_r+1}}\cdots\frac{p^{\lambda}-p^{\lambda-1}}{p^{\lambda}-p^{\lambda-1}}.$$

* G. A. Miller, *Amer. Jour. of Mathematics*, vol. 27 (1905), p. 16.

This formula may evidently be reduced to the following much simpler form:

$$(p^{\lambda_1}-1)(p^{\lambda_1-1}-1)\cdots(p-1)(p^{\lambda_2}-1)(p^{\lambda_2-1}-1)$$
$$\cdots(p-1)\cdots(p^{\lambda_r}-1)(p^{\lambda_r-1}-1)\cdots(p-1)\div(p-1)^{\lambda}.$$

From the known theorem that the order of the group of isomorphisms of any abelian group is equal to the number of ways in which a set of independent generators may be selected, it results directly that the order of I is equal to the following continued product:

$$(p^{m}-p^{m-\lambda_1})(p^{m}-p^{m-\lambda_1+1})\cdots(p^{m}-p^{m-1})(p^{m-\lambda_1(m_1-m_2)}$$
$$-p^{m-\lambda_1(m_1-m_2)-\lambda_2})$$
$$p^{m-\lambda_1(m_1-m_2)}-p^{m-\lambda_1(m_1-m_2)-\lambda_2+1})\cdots(p^{m-\lambda_1(m_1-m_2)}-p^{m-\lambda_1(m_1-m_2)-1})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(p^{\lambda m_r}-p^{\lambda m_r-\lambda_r})(p^{\lambda m_r}-p^{\lambda m_r-\lambda_r+1})\cdots(p^{\lambda m_r}-p^{\lambda m_r-1}).$$

Hence a simple formula for the order of I is as follows:

$$(p^{\lambda_1}-1)(p^{\lambda_1-1}-1)\cdots(p-1)(p^{\lambda_2}-1)(p^{\lambda_2-1}-1)\cdots$$
$$(p-1)(p^{\lambda_r}-1)(p^{\lambda_r-1}-1)\cdots(p-1)p^{m'},$$

m' being equal to the following double sum:

$$\sum_{\alpha=1}^{\alpha=r}\sum_{\beta=1}^{\beta=\lambda_\alpha} m-\lambda_1(m_1-m_\alpha)-\lambda_2(m_2-m_\alpha)-\cdots-\lambda_{\alpha-1}(m_{\alpha-1}-m_\alpha)-\beta.$$

From the value of the order of I and the formula giving the number of the Sylow subgroups of order $p^{m'}$ contained in I, it results directly that each of these Sylow subgroups is transformed into itself by exactly $p^{m'}(p-1)^{\lambda}$ operators of I. Hence we have established the following theorem:

If an abelian group of order p^m has exactly λ invariants, the Sylow subgroup of order $p^{m'}$ in the group of isomorphisms of this abelian group is transformed into itself under this group of isomorphisms by a subgroup of order $p^{m'}(p-1)^{\lambda}$.

A direct corollary from this theorem is as follows:

A necessary and sufficient condition that the Sylow subgroups of order $p^{m'}$ in the group of isomorphisms of an abelian group of order p^m are transformed into themselves only by their own operators under this group of isomorphisms is that $p=2$.

As a very special case of the formula giving the number of Sylow subgroups of order $p^{m'}$ in I, there results the following known theorem: A necessary and sufficient condition that the group of isomorphisms of an abelian group of order p^m contains only one Sylow subgroup of order $p^{m'}$ is that it contains no two equal invariants. It also results directly from this formula that a necessary and sufficient condition that an abelian group of order p^m has a group of isomorphisms whose order is of the form $p^{m'}$ is that $p = 2$ and that no two invariants of this abelian group are equal to each other.

Each of the given groups of order $p^{m'}(p-1)^\lambda$ which involve only one subgroup of order $p^{m'}$ includes $p-1$ of the $p^{m_1-1}(p-1)$ invariant operators of I. It also involves a subgroup of order $(p-1)^\lambda$, which is generated by λ operators of order $p-1$ such that each of these operators is commutative with all the independent generators of G except one, in a given set of independent generators, and that no two of these λ operators are non-commutative with the same independent generator. Hence this subgroup of order $(p-1)^\lambda$ is abelian and it is the direct product of λ cyclic groups of order $p-1$. It transforms into itself the cyclic subgroups generated by one, and only one, set of independent generators of G, and hence to each pair of Sylow subgroups of order $p^{m'}$ contained in I there correspond two sets of abelian subgroups of order $(p-1)^\lambda$ such that none of these subgroups is common to the two sets. In particular, we have established the following theorem:

If an abelian group of order p^m has λ invariants, each of the Sylow subgroups of order $p^{m'}$ in the group of isomorphisms of this abelian group is transformed into itself by a group of order $(p-1)^\lambda$, which is the direct product of λ cyclic groups of order $p-1$.

Denoting by I' a Sylow subgroup of order $p^{m'}$ contained in I, we proceed to determine some restrictions on the orders of the operators of I'. It is well known that all these orders are divisors of p^{m-1}. Whenever G involves more than one invariant which is equal to p^{m_1} it is easy to prove that I' must involve at least one operator of order p^{m_1}. In fact, if S_1 and S_2 represent two independent generators of G and if each of these generators is of order p^{m_1}, I' includes an operator t which is commutative with each one of a set of independent generators of G, except S_2, and transforms S_2 into S_1S_2. As t^n transforms S_2 into $S_1{}^nS_2$, the order of t is clearly equal to p^{m_1}.

We proceed to prove that I' does not involve any operator whose order exceeds p^{m-1} whenever G does not involve more than p invariants which are equal to the same number. In fact, this result follows almost immediately from a method which has been employed to find the orders of operators in the group of isomorphisms of an abelian group.* According to this method the pth power of an operator of I' transforms every independent generator of G into itself multiplied by an operator whose order is less than the order of this independent generator. Hence there results the theorem:

If an abelian group of order p^m does not involve more than p invariants which are equal to each other, the group of isomorphisms of this abelian group involves no operator whose order is a power of p and exceeds the largest invariant of this abelian group.

UNIVERSITY OF ILLINOIS.

THEORY OF PRIME NUMBERS.

Handbuch der Lehre von der Verteilung der Primzahlen. Von Dr. EDMUND LANDAU, ordentlichem Professor der Mathematik an der königlichen Georg-August-Universität zu Göttingen. Vol. I, pp. xviii + 1–564; Vol. II, pp. ix + 565–961. Leipzig und Berlin, B. G. Teubner, 1909.

THE central problem in the prime number theory consists in proving that the number $\pi(x)$ of primes less than or equal to x may be represented asymptotically by the expression $x/\log x$, or

$$\lim_{x=\infty} \frac{\pi(x) \log x}{x} = 1. \tag{1}$$

This formula was conjectured by Legendre and Gauss at the end of the eighteenth century, but the first definite step toward a proof was taken by Tchebychef, who showed in 1851–52 that for sufficiently large values of x the quotient $\pi(x) \log x/x$ lies between two positive boundaries, one of them less and the other greater than unity. The next great advance is marked by Riemann's paper of 1859 "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse." His point of departure is the equation established by Euler

* G. A. Miller, BULLETIN, vol. 7 (1901), p. 351.

$$(2) \qquad \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}} \quad \text{for } s \text{ real and} > 1,$$

where the right-hand product extends over all prime numbers p. The occurrence of all prime numbers in (2) suggests an intimate relation between the prime number problem and the function $\zeta(s)$ (the Riemann Zeta function) defined for $s > 1$ by the left-hand member of (2). Riemann's fundamental idea consists in the introduction of complex values of s and the investigation of the properties of the analytic function $\zeta(s)$ of the complex variable s; in this direction, he gives a number of important and profound theorems, part of which, however, are incompletely proved or even stated as mere conjectures. While Riemann does not arrive at a proof of (1), his function-theoretic point of view is the one which has made further progress possible. The difficulties presented by the unproven statements of Riemann were, however, so great that no further advance was made until 1893, when Hadamard, by his theory of entire functions of finite rank, created methods capable of proving several of Riemann's assertions and prepared the way for the proof of (1), which was accomplished independently by Hadamard and de la Vallée Poussin in 1895. Since then, the prime number theory has been enriched in results, as well as simplified in methods, by an ever increasing number of mathematicians, until the literature on the subject has now grown to such an extent as to necessitate a systematic presentation of its main results.

None could be more competent to meet this need than Professor Landau, to whom the greater part of recent progress in the theory of primes is due. In his exposition of a theory presenting as many open questions as the present one, Professor Landau has followed what is probably the most commendable course, that of dividing the results, according to the methods required to establish them, into four large groups, namely those obtainable, *first*, by quite elementary methods, *second*, by analysis in a real variable, *third*, by elementary processes in the theory of analytic functions of a complex variable, and *fourth*, by Hadamard's theory of entire functions of finite rank. This treatment of the subject has two great advantages: it enables the author to determine as nearly as possible just how far each of these four methods will lead,

which is of invaluable service to those engaged in research work, and furthermore, it affords an opportunity of introducing all simplifications of which the frequently long and difficult original investigations have shown themselves capable, without slighting the historical aspect of the subject. Instead of burdening the text with footnotes, the author has assembled all bibliographical references at the end of the second volume under the heading "Quellenangaben," pages 883–907, followed by a very complete "Literaturverzeichnis," pages 908–961.

The only prerequisite for the study of the work under review is a good working knowledge of the elements of the theory of functions of a complex variable, and particularly of Cauchy's theorem; whatever results from the elementary theory of numbers or Hadamard's theory of entire functions are needed, are presented fully and in a very accessible form in the text.

After an introduction on the history of the prime number problem (pages 3–55), the work is divided into five books and twenty-seven sections.

The first book (sections 1–4) treats of the prime number problem, i. e., the asymptotic expression of $\pi(x)$. Section 1 (pages 59–102) deals with the results obtainable by quite elementary methods. First the following very convenient notations are introduced, x being positive but not necessarily an integer:

$$f(x) = O(g(x))$$

signifies that two positive constants ξ and A exist such that $|f(x)| < Ag(x)$ for $x \geqq \xi$ (examples: $\sqrt{x} = O(x)$, $e^{ix} = O(1)$);

$$f(x) = o(g(x))$$

means that

$$\lim_{x=+\infty} \frac{f(x)}{g(x)} = 0 \quad \text{(examples: } \log x = o(\sqrt{x}), \frac{1}{x} = o(1)\text{)};$$

denoting by $[x]$ the greatest integer contained in x, the sum

$$\sum_{n=v}^{w} f(n)$$

stands for a sum extended over all integers n from $[v]$ inclusive to $[w]$ inclusive; and finally

$$\sum_{p \leqq x} f(p), \quad \prod_{p \leqq x} f(p)$$

signify that the sum, or product, is extended over all primes p not exceeding x.

Now the functions

$$T(x) = \sum_{n=1}^{x} \log n, \quad \vartheta(x) = \sum_{p \leqq x} \log p,$$
$$\psi(x) = \vartheta(x) + \vartheta(\sqrt{x}) + \vartheta(\sqrt[3]{x}) + \cdots = \sum_{p^m \leqq x} \log p \tag{3}$$

are introduced, and the Tchebychef-de Polignac identity

$$T(x) = \sum_{n=1}^{x} \psi\left(\frac{x}{n}\right) \tag{4}$$

is proved. From the definition of $T(x)$, it follows that

$$T(x) = x \log x - x + O(\log x); \tag{5}$$

the combination of (4) and (5) shows that

$$\limsup_{x=\infty} \frac{\vartheta(x)}{x} = \limsup_{x=\infty} \frac{\psi(x)}{x} \leqq 2 \log 2,$$
$$\liminf_{x=\infty} \frac{\vartheta(x)}{x} = \liminf_{x=\infty} \frac{\psi(x)}{x} \geqq \log 2,$$

whence the conclusion

$$\frac{\pi(x) \log x}{x} - \frac{\vartheta(x)}{x} = O\left(\frac{1}{\log x}\right) \tag{6}$$

is drawn, showing that both terms on the left side have the same lim. sup. and the same lim. inf. These upper and lower boundaries are now narrowed down step by step, and the interval between them is shown to contain unity as an interior point. Tchebychef's proof of Bertrand's celebrated postulate (for $x \geqq 7$ there exists at least one prime p such that $\frac{1}{2}x < p \leqq x - 2$) is given, as well as two asymptotic formulas due to Mertens:

$$\sum_{p \leqq x} \frac{\log p}{p} = \log x + O(1),$$
$$\sum_{p \leqq x} \frac{1}{p} = \log \log x + B + O\left(\frac{1}{\log x}\right), \tag{7}$$

B being a numerical constant.

In section two (pages 103–150), the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{8}$$

are introduced, and some of their elementary properties as functions of the *real* variable s are given. The application of these results to the special Dirichlet series for the functions $\zeta(s)$ and $\zeta'(s)/\zeta(s)$ gives new and simpler proofs for the theorems of the first section, but does not lead much farther; in particular, these methods are insufficient to prove the asymptotic relation (1). For this problem, the introduction of complex values of the variable $s = \sigma + ti$ in section three (pages 151–269) proves more efficient. After defining $\zeta(s)$ as the analytic function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for } \sigma > 1,$$

it is shown that equation (2) is true for these values of s, and that consequently $\zeta(s)$ has no zeros in the half plane $\sigma > 1$. Two different proofs are given for the fact that $\zeta(s) - 1/(s-1)$ is holomorphic for $\sigma > 0$. By an ingenious artifice due to Mertens, it is shown that $\zeta(s)$ has no zeros to the right of the curve defined by

$$\sigma = 1 - \frac{1}{c(\log|t|)^9} \text{ for } |t| \geqq 3, \quad \sigma = 1 - \frac{1}{c(\log 3)^9} \text{ for } |t| \leqq 3, \tag{9}$$

and that, to the right of this curve and for $|t| \geqq 3$,

$$\begin{gathered} \frac{1}{c_2(\log|t|)^7} < |\zeta(s)| < c_1 \log|t|, \\ \left|\frac{\zeta'(s)}{\zeta(s)}\right| < c_3(\log|t|)^9, \end{gathered} \tag{10}$$

c, c_1, c_2 being positive constants, as well as all c's with subscripts that are introduced later.

Defining $\Lambda(n)$ as the numerical function which is zero for $n = 1$ and for all values of n which are not powers of a prime, but equals unity when n is a power of a prime, the formula

$$\sum_{n=1}^{x} \Lambda(n) \log \frac{x}{n} = -\frac{1}{2\pi i} \int_{2-xi}^{2+xi} \frac{x^s}{s^2} \frac{\zeta'(s)}{\zeta(s)}\, ds + O(1)$$

is established, and by the application of Cauchy's theorem to the integral

$$\int \frac{x^s}{s^2}\frac{\zeta'(s)}{\zeta(s)}\,ds \tag{11}$$

taken over the closed contour bounded by the curve (9) and the three straight lines $t = x^2$, $\sigma = 2$ and $t = -x^2$, it follows that

$$\sum_{n=1}^{x}\Lambda(n)\log\frac{x}{n} = x + O(xe^{-\sqrt[12]{\log x}}). \tag{12}$$

By elementary transformations of (12), it is now shown that

$$\begin{aligned}\vartheta(x) &= x + O(xe^{-\sqrt[12]{\log x}}),\\ \pi(x) &= \int_2^x \frac{du}{\log u} + O(xe^{-\sqrt[12]{\log x}}),\end{aligned} \tag{13}$$

where the second formula includes the prime number theorem (1) as a very special case.

The formulas (13) furnish the means for obtaining asymptotic expressions for various sums of the type $\Sigma_{p\leq x}F(p, x)$, where the function $F(p, x)$ is subject to certain very general conditions. It is shown that there are more primes between 1 and x than between x and $2x$ for x sufficiently large; the formulas

$$\liminf_{x=\infty}\frac{\varphi(x)}{\dfrac{x}{\log\log x}} = e^{-C}, \qquad \limsup_{x=\infty}\frac{\log\tau(x)}{\dfrac{\log x}{\log\log x}} = \log 2,$$

are proved for integral values of x, $\varphi(x)$ being the number of integers $\leq x$ and relative primes to x, C Euler's constant, and $\tau(x)$ the number of divisors of x; various other applications of (13) are also given.

Section four (pages 270–388) goes deeper into the properties of the analytic function $\zeta(s)$. It is shown according to Riemann that $\zeta(s) - \dfrac{1}{s-1}$ is an entire function of s, and that

$$\xi(s) = \frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s)$$

satisfies the functional equation

$$\xi(1-s) = \xi(s). \tag{14}$$

The principal results of Hadamard's theory of entire functions of finite rank are now exposed in the particular case where the rank equals zero, and it is shown that, writing $\xi(s) = g(x)$ where $s = \frac{1}{2} + i\sqrt{x}$, $g(x)$ is an entire function of rank zero, so that, $\xi_1, \xi_2, \cdots$ being its zeros, the relation

$$g(x) = g(0) \prod_{\nu=1}^{\infty} \left(1 - \frac{x}{\xi_\nu}\right)$$

subsists. From this formula it follows that

$$(s-1)\zeta(s) = \tfrac{1}{2} e^{bs} \frac{1}{\Gamma\left(\frac{s}{2}+1\right)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}, \tag{15}$$

b being a numerical constant and the ρ's being all those zeros of $\zeta(s)$ which have their real parts between 0 and 1. In consequence of (15), it is shown that the curve (9), to the right of which $\zeta(s)$ has no zeros, may be replaced by the curve

$$\sigma = 1 - \frac{1}{a \log |t|} \text{ for } |t| \geqq 2, \quad \sigma = 1 - \frac{1}{a \log 2} \text{ for } |t| \leqq 2, \tag{16}$$

a being a positive constant, and an application of Cauchy's theorem similar to the one leading to (13) now gives

$$\pi(x) = \int_2^x \frac{du}{\log u} + O(xe^{-a\sqrt{\log x}}), \tag{17}$$

where $\alpha \geqq \frac{1}{5}$. Furthermore, two conjectures of Riemann are proved: one concerning a certain series involving the ρ's, and another stating that the number of zeros $\rho = \beta + \gamma i$ satisfying the condition $0 < \gamma \leqq T$ is equal to

$$\frac{1}{2\pi} T \log T - \frac{1 + \log(2\pi)}{2\pi} T + O(\log T). \tag{18}$$

The last of Riemann's conjectures is that all the ρ's have their real part $\beta = \frac{1}{2}$, and the question of proving (or disproving) it is the most important unsolved problem in the theory of primes. It is intimately connected with the question of reducing the order of the remainder term in (17). Section

four ends with the demonstration that, if this conjecture of Riemann is true, (17) may be replaced by

$$\pi(x) = \int_2^x \frac{du}{\log u} + O(\sqrt{x}\log x).$$

Book two (sections 5 to 8, pages 391–564) deals with the prime numbers contained in an arithmetical progression $ky + l$ ($y = 0, 1, 2, \cdots$), where l is any one of the $h = \varphi(k)$ integers not exceeding k and relative primes to it. In section five, the properties of the Dirichlet characters $\chi_k(n)$ ($k = 1, 2, \cdots, h$) (which are either zero or certain roots of unity, depending upon the nature of the decomposition of n in prime factors) are developed, and the Dirichlet series

$$L_k(s) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s} \quad (k = 1, 2, \cdots, h) \tag{19}$$

are introduced, which generalize the Zeta function to the present case. The consideration of the series (19) for real values of s leads to Dirichlet's celebrated theorem: every arithmetical progression $ky + l$, where l and k are relative primes, contains an infinity of prime numbers.

In sections six and seven, the introduction of a complex variable s in (19) leads to results quite similar to those obtained for $\zeta(s)$ in book one, and culminating in the theorem: the number $\Pi(x)$ of primes of the form $ky + l$ and not exceeding x is asymptotically

$$\Pi(x) = \frac{1}{h}\int_2^x \frac{du}{\log u} + O(xe^{-a\sqrt{\log x}}). \tag{20}$$

In section eight, various applications of this result are given, for instance the theorem: every positive integer above a certain limit may be decomposed into a sum of not more than eight positive cubes.

The second volume begins with book three entitled "the function $\mu(n)$ and the distribution of numbers without quadratic divisors" (sections 9–13, pages 567–621). The Möbius function $\mu(n)$ equals unity for $n = 1$, zero when n has a quadratic divisor, and $(-1)^\rho$ when n is the product of ρ different primes. In this book, some new properties of the Zeta function are deduced, which lead to the formulas

$$\sum_{n=1}^{x}\mu(n) = O(xe^{-a\sqrt{\log x}}), \qquad \sum_{n=1}^{x}\frac{\mu(n)}{n} = O(e^{-a\sqrt{\log x}}),$$

(21)

$$\sum_{n=1}^{x}\frac{\mu(n)\log n}{n} = -1 + O(e^{-a\sqrt{\log x}}).$$

Book four (sections 14–16, pp. 625–639) treats similar problems concerning the numbers without quadratic divisors contained in an arithmetic progression, and book five (sections 17–20, pages 641–719) contains convergence proofs for various types of series connected with prime numbers, as well as some theorems on $\pi(x)$, of which the following is an example: above any finite limit however large, there exist numbers x for which

$$\pi(x) - \int_2^x \frac{du}{\log u} + \frac{1}{2}\int_2^{\sqrt{x}} \frac{du}{\log u} > \frac{1}{c}\frac{\sqrt{x}}{\log x},$$

as well as numbers x for which the left-hand member is less than $-\frac{1}{c}\frac{\sqrt{x}}{\log x}$.

Finally book six (sections 21–27, pages 723–882) contains an elaborate investigation, with many applications to the Zeta function, of functions of the complex variable s defined by a generalized Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

where the λ_n are real and increase monotonely toward infinity with n.

In concluding this review of Professor Landau's monumental work, it should be stated that the exposition is a model of clearness and rigor. There are surprisingly few misprints, none of them serious, and the make-up of the book even surpasses the usual high standard of the Teubners.

T. H. GRONWALL.

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THE PRICE LEVEL.

The Purchasing Power of Money. Its Determination and Relation to Credit, Interest and Crises. By IRVING FISHER. Assisted by HARRY G. BROWN. New York, The Macmillan Company, 1911. xxii+505 pages.*

A LONG-CONTINUED and extensive rise or fall in the general level of prices brings on a serious disarrangement of the status of parties to long time contracts involving payments of fixed or nearly fixed sums of money, unbalances the economic equilibrium which is never too stable, and develops in its wake various extreme political doctrines. The extended decline of prices from the seventies to the nineties of the past century worked hardship to those who had borrowed money in the earlier part of the period and had to repay the money with interest during the latter part of the period when their produce brought a reduced sum of money per unit. Hence populism and free silver. The succeeding great rise in prices, which has now run for nearly twenty years with no immediate prospect of cessation, is working a like hardship on the wage-earning and salaried classes, whose income does not rise nearly so fast as the prices of the articles they consume. Hence socialism, the single tax, and the like.

It would work for greater justice and contentment all around if the general level of prices could be kept more or less constant. Moreover, changing prices give to the brighter, to the more wide-awake, often to the merely luckier members of society an advantage for profit over their more stupid or unfortunate brethren. This is distinctly undemocratic; for, as Emile Faguet so brilliantly points out,† the worship of democracy is the cult of incompetence. It is an old-fashioned, unprogressive idea, perhaps, but still an idea widely held by thoughtful persons that prices cannot be maintained for any long period in defiance of economic laws, that on the contrary the desirable constancy of the level of prices can only be worked out by a painstaking study of and compliance with those laws.

* For a review of the earlier works, Capital and Income, and Rate of Interest, written in a similar spirit and helpful, though not necessary, to the understanding of the present work, see this BULLETIN, vol. 15, p. 169.

† The Cult of Incompetence, New York, Dutton, 1912.

In the Purchasing Power of Money, Fisher makes a detailed, systematic, scientific study of the causes affecting the general price level, and with this as a basis he makes some tentative suggestions, which he has modified and worked out in greater detail in more recent monographs, for the stabilization of the purchasing power of money. These suggestions have attracted widespread attention and comment;* they have been cordially applauded and seriously combatted, often, we daresay, by persons who have neglected the main part of the author's text; but we shall pass them over and, indeed, confine our attention chiefly to one point of mathematical interest in the volume.†

To get a quantitative basis for the treatment of prices we may equate expenditures and the money-value of goods bought. Thus

$$MV + M'V' = PT, \tag{1}$$

where M is the quantity of money in circulation and V the velocity of circulation (MV giving expenditures in cash), M' the quantity of deposits subject to check and V' their velocity of circulation ($M'V'$) giving the total expenditure by check), T the total amount of trade, and P the average price. Certain general conclusions follow from this equation. For instance, if the amount of trade increases, while M, V, M', V' remain constant, the price level must fall; if M and T increase proportionately, while V and V' remain constant, P will decrease or increase according as M' increases more slowly or more rapidly than M and T.

Various consequences are readily obtained from the equation of exchange (1), but the determination of the equation itself is not so easy as it might look to a careless thinker. The difficulties lie in the fact that P and T individually are quite indeterminate. An average price level P means nothing until the rules for obtaining the average are specified, and independent rules for evaluating P and T may not satisfy (1). For instance suppose sugar is 5c. a pound, bacon 20c. a pound, coffee 35c. a pound. The average price is 20c. If a person buys 10 lbs. of sugar, 3 lbs. of bacon, and 1 lb. of coffee, the total trading is in 14 lbs. of goods. The total expenditures is $1.45;

*See, for example, recent numbers of the *American Economic Review*.
†For a review of some other aspects see *Science*, May 16, 1913, p. 758.

the product of the average price by the total trade is $2.80; the equation of exchange is very far from satisfied.

After thinking this example over one might come to the conclusion that nobody would use such a crude method of averaging prices. The method is, however, the simplest and most obvious and is widely used. Bradstreet's much quoted index of 96 commodities is (see page 417) an average of prices per pound; relatively small and unimportant changes in the prices of expensive articles may entirely mask relatively large movements in the prices of cheap articles. Despite the crudeness of the average for P it is possible, of course, to satisfy the equation of exchange by defining the measure of T as

$$T = \frac{MV + M'V'}{P}.$$

In the above example with an average price per pound of 20c. the total trade would be $7\frac{1}{4}$ lbs. If the trade were measured as 14 lbs., the average price would have to be about $10\frac{1}{3}$c.

The difficulty of reconciling with equation (1) an average price and a total trade, each evaluated independently by natural methods, is great and requires a searching analysis of what is meant by an average and what is really natural and reasonable. There are other difficulties. One of them I came across a few years ago when trying to help a friend make a stock-average which would be representative of prices on the New York Exchange from 1872 to date. It was impossible to find ten stocks that had remained active and representative over so long a period, and the initial list of ten had to be modified by several successive changes. In the same way if any average of commodity prices contains articles which go out of consumption entirely or to any very great extent, there is a discontinuity introduced in P, and a good index P should satisfy the condition of allowing one article to be replaced by another.

The mathematical interest in Fisher's work lies chiefly in the discussion of the theory of averages with especially reference to the determination of the best form of index number for prices and trade. This is applied, not pure, mathematics; the analysis aids in forming the judgment but is not determinative, for no index number is found to satisfy all the tests which are desired. Forty-four different indices are comparatively dis-

cussed. The treatment of these matters is for the most part relegated to a long appendix to Chapter X. In itself it constitutes a notable contribution to political arithmetic.

The general course of the text may be summarized as follows. After some preliminary definitions (Chapter I), the author treats the simple case $MV = PT$ of the equation of exchange which would arise under a financial system with no checking accounts, and determines particularly the relation of the quantity of money M to the price level P. He then (Chapter III) generalizes so as to include the term $M'V'$ and the effect of deposit currency. At the close of the chapter he has come to the conclusion that normally the available currency is apportioned between M and M' in a certain more or less definite, even though elastic, ratio. This conclusion is the result of forming a judgment, it is not deductively obtained; the statistics given for this country from 1896 to date show a progressive increase of the ratio M'/M, and recent statistics for other countries would probably show the same tendency. We can retain the conclusion only by assuming that for a couple of decades conditions have not been normal. The advantage of including the constancy of M'/M among our axioms is that we thereby restore theoretically and normally the proportionality between M and P which has for all practical purposes been entirely upset by the recent great increase in the ratio $M'V'/MV$.

In Chapter IV transition periods are analyzed; it is here, chiefly, that justification for the subtitle referring to credit, interest and crises, may be found. The next two chapters analyze a number of influences which indirectly affect the purchasing power, and Chapter VII deals with the influence of monetary systems. The eighth chapter, on the effect of the quantity of money and other factors on purchasing power and on each other, touches many topics but is chiefly directed toward showing (1°) that the price level P is the passive element, the effect, the dependent variable in the equation of exchange and not the cause, and (2°) that the special quantity theory that variations in M *normally* produce proportional changes in P is true.

There are so many bitter and blind opponents to any specific theory (and the quantity theory is no exception) that an author who has to qualify the fundamental proposition of his theory by so vague a restriction as "normally" might do well to

use the term hypothesis in place of theory; and he can certainly not spend too much pains on separating those portions of his work which depend upon such a hypothesis from those which do not.

Chapter IX shows the necessity for an index of prices, and Chapter X, aided by the long mathematical appendix previously reviewed, seeks after the best index. Statistical matter bearing on the previous text now fills two chapters, and the text closes with the discussion of the possibility of keeping the general price level more nearly constant. There are numerous appendices and a very full index.

To the mathematician Fisher's work always appears more sympathetic than that of many of his economic colleagues by virtue of its greater respect for logical and numerical values. This latest book is no exception.

EDWIN BIDWELL WILSON.

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SHORTER NOTICES.

Vorlesungen über Geometrie, mit besonderer Benutzung der Vorträge von ALFRED CLEBSCH, bearbeitet und herausgegeben von DR. FERDINAND LINDEMANN. Zweite, vermehrte Auflage. Leipzig, Teubner. I Band, I Teil, 1 Lieferung, 1906, vi + 480 pp.; 2 Lieferung, 1910, 288 pp.

THIS revised edition of Clebsch's inimitable and classical lectures is much more than a reprint of the first edition which appeared in 1875–76. Much has been done since then to enrich the subject, and Dr. Lindemann has done well to include the most important of these researches in this revision. The extent of the additions to the text may be seen from a comparison of the number of pages for Part I; this includes the same general subjects in the two editions and yet covers about three times as many pages in the second as in the first (the size of the type and the page differ only slightly). If this ratio is upheld in the other parts, Volume I will contain about 3,000 pages of text.

It is unfortunate that four years should elapse between the two installments. Upon the appearance of the first installment, the completion of Part I was promised within a few

months. Even this second installment does not complete Part I, and three years have elapsed since this appeared. At this rate no time limit can be placed upon the appearance of Parts II and III. The custom of sending forth an installment which stops short in the middle of a sentence of some exposition or proof of a theorem is also unfortunate; its macks too much of the serial installments of some thrilling novel, all the more so when one has to wait years instead of merely a day, a week, or a month for the completion of that sentence.

The value of the book is greatly enhanced by the copious footnotes. These give very complete references and brief descriptions of the work done in the subject both before and since Clebsch's time. They also tell, for each chapter, just what parts of the text are taken from the original Clebsch lectures, and which parts have been added for the revised edition.

We shall briefly indicate the more important parts of the text which have been added. Part I is divided into three sections. The first section, entitled Introductory considerations.—Point ranges and pencils of rays, gives the projective analytic geometry of the range and the pencil. Here a few more illustrative examples are inserted and some special cases of general theorems are treated in greater detail. Chapter VII, treating of the general analytic form for projective relations, and Chapter VIII, containing a large number of theorems on medians of quadrilaterals and pentagons, are new.

The second section is entitled The curves of the second order and the second class. Here there have been added six chapters treating respectively of: (VII) the classification of conics according to the form of the projective transformation which generates the same; (XIII) the complete Pascal hexagon with its related Steiner and Kirkman points; (XIV) polar relations and the Pascal hexagon, including the work of Bauer on the ten conics associated with the six points on a conic; (XV) theorems on the application of general analytic methods to conics and quadrilaterals; (XVI) the geometry of the triangle with its Feuerbach circle (following the work of Salmon-Fiedler); (XVII) imaginary elements, giving a deeper insight into the geometric "Sinn" of the imaginary elements according to the beautiful theory of von Staudt (the representation, in the real domain, of two conjugate imaginary points by the involution of which these are the double elements);

and thus all projective results in analytic geometry of the plane in which imaginary elements occur may be interpreted by means of real geometric elements; the exposition in this chapter follows the work of Lüroth and Stolz.

The third section is entitled Introduction to the theory of algebraic forms. Here we find the most extensive additions. The discussion of the identical relations existing between the discriminants of two binary quadratic or cubic forms is extended to the case of n such forms. The other important additions are contained in the last thirteen chapters, none of which had a place in the first edition. These deal respectively with: (X) the representation of binary forms upon a conic, which serves to geometrize and thus classify and emphasize the relations existing among the invariants and covariants of binary forms; (XI) the collineations of a conic into itself and the finite groups of such transformations; (XII) the "tenfold" Brianchon hexagon and the collineations of the related conic; (XIII) polygons and binary forms with their application to the equations of the fifth degree; (XIV) the expansion of binary forms into series (according to the work of Clebsch and Gordan), which serves as a unifying principle between the general methods of the text (using the Clebsch-Aronhold notation) and the non-symbolic and non-homogeneous methods employed to solve special problems; (XV–XVIII) the typical or canonical representation of binary forms by means of partial differential equations for invariants and covariants (according to Clebsch and Gordan), which serves to give an insight into the invariant theory of binary forms of higher order; this canonical representation also gives a method for determining a finite system of invariant forms by means of which all invariants and covariants of a given form may be constructed by rational processes, and finally leads to the proof that the system of invariant forms of a binary algebraic form is finite; (XIX–XXI) the simplest invariants and covariants of a binary form of the fifth and sixth orders, and geometric and arithmetic applications of these to the pentagon and the equation of fifth degree; (XXII) some problems in the theory of binary forms of higher order (only one page of this is printed in the second installment).

Part I is to be completed by a discussion of the representation of binary forms in the complex plane, and an application of this and of cubic forms to the geometry of the triangle and its remarkable points.

Parts II and III are to cover the general theory of algebraic curves, the curves of the third order and the third class, the geometry of an algebraic curve and its relation to the theory of abelian integrals, and a study of the connex.

Dr. Lindemann has done and is doing a monumental work which deserves to rank among the highest in works on geometry; let us hope for its early completion.

JOSEPH LIPKA.

Die Idee der Riemannschen Fläche. Mathematische Vorlesungen an der Universität Göttingen, no. V. By HERMANN WEYL. Leipzig, Teubner, 1913. x+166 pp. 8 M.

THE volume under review is the fifth of the series of Göttingen Lectures on mathematics, the first four of which were given respectively by Klein, Minkowski, Voigt, and Poincaré. It is announced that four more are under preparation by Runge, Schwarzschild, Toeplitz, and Wiechert.

Before speaking specifically of the work of Weyl, I should like to express my great appreciation of the plan of publishing lectures of this sort, in which the lecturer gives a limited subject mature thought, brings it thoroughly up to date, and perhaps adds contributions of his own. This is a period of what might be called "frenzied" research, in which there is a tendency to rush into print every time a new result is reached. While the intense activity is altogether praiseworthy, it results in a congestion of material in the mathematical journals, and a flood of papers so great that it is impossible for an individual to discover what is important and of interest to him. If the basic ideas of mathematics were shifting as fast as those of some other sciences, the editors of mathematical journals might find themselves in the position of an editor who recently said that his journal was so far behind the material offered it for publication that the authors changed their ideas and wished to withdraw their papers before they could appear in print. Even if we grant that, on the whole, it is for the best that fragmentary investigations in almost unlimited number should continue to be published, the more formal discussions, in which the center of interest is in the subject and not alone in the greater or lesser part which may be new, are of great value and should be encouraged. In this connection it is gratifying that the American Mathematical Society has adopted the policy of publishing the Colloquium Lectures.

They should become a strong factor in the development of mathematics in this country.

The lectures by Weyl aim at two things, viz., laying with complete rigor the foundations for the construction of Riemann surfaces, and then showing their usefulness by considering on them various integrals, the Riemann-Roch theorem, Abel's theorem, and the problem of uniformizing functions. These lectures owe much to Klein (they are dedicated to him) both because of the fundamental work he has done on the subject and also because they were given where his influence is so great; but Weyl does not shine only by reflected light, for there is evidence in the plan, the choice of material, and the exposition that he is a master of the subject on which he writes, and his task is splendidly carried out.

There is probably no other part of the theory of functions where on the whole the logic is so much slurred over as in the theory of Riemann surfaces. Simple examples are given, and then the simplicity and intuitional character of the ideas make it natural to abandon the arithmetic character of the proofs which ordinarily prevail elsewhere. It is interesting to find Weyl starting out at the very beginning with the Weierstrass definition of an analytic function, defining then an "analytisches Gebilde" by means of a parametric representation of the functional relation in question, and then defining a surface by ascribing appropriate properties to a two-dimensional manifold and showing it may be considered as an analytisches Gebilde. This prepares the way for the definition of a Riemann surface by means of analysis situs considerations. Then follow the definitions of properties and quantities (e. g., the genus) associated with Riemann surfaces, the method of construction of a Riemann surface of canonical form, and the proof of a series of theorems such as that every Riemann surface has two sides. In this discussion interesting connection is made with some recent general results of Brouwer. The whole first part seems to be exceptionally well done both in mathematical quality and in method of presentation.

The second part of Weyl's lectures contains various applications to functions on Riemann surfaces. The distinctive features of this part are less conspicuous than in the first part, but the standard important problems are well treated. Especially interesting is the discussion of uniformizing functions, a subject whose development is rather recent and which com-

pletes in a sense the conception of Weierstrass of the analytisches Gebilde by employing a suitable parameter of position on the whole Riemann surface.

F. R. MOULTON.

NOTES.

AT the meeting of the London mathematical society held February 12 the following papers were read. By G. T. BENNETT, "Exhibition and explanation of some models illustrating kinematics"; by H. M. MACDONALD, "Formulae for the spherical harmonic $P_n - m(\mu)$, when $1 - \mu$ is a small quantity"; by E. W. HOBSON, "The representation of the symmetrical nucleus of a linear integral equation"; by W. F. SHEPPARD, "Fitting of polynomials by the method of least squares" (second paper); by H. BATEMAN, "The differential geometry of point transformations between two planes"; by M. MCKENDRICK, "Studies in the theory of continuous probabilities."

THE thirtieth anniversary of the foundation of the Circolo matematico di Palermo will be celebrated with appropriate ceremonies on Tuesday, April 14, in the aula of the University of Palermo. On this occasion a gold medal, provided by members' subscription, will be presented to Professor G. B. GUCCIA in recognition of his services as founder of the society and director of the *Rendiconti*. The Circolo has now nearly one thousand members, of whom it may be noted that one hundred and fifty are Americans. Thirty-seven volumes of the *Rendiconti* have been published.

THE tenth annual meeting of the Association of Ohio teachers of mathematics and science was held at Ohio State University on April 3–4. The programme included addresses by Mr. J. F. BARKER on technical training in the schools and by Professor H. E. SLAUGHT on the final report of the committee of fifteen on a geometry syllabus.

CAMBRIDGE UNIVERSITY.—The following mathematical courses are announced for the Easter term:—By Professor E. W. HOBSON: History of the invention of the calculus, three

hours.—By Professor H. F. BAKER: Existence theorems of automorphic functions, two hours.—By Mr. T. S. HERMAN: Differential geometry, three hours; Hydrodynamics, three hours.—By Dr. J. A. BROMWICH: Diffraction of electric waves, three hours.—By Mr. A. C. GRACE: Elements of Fourier analysis and calculus of variations, two hours.—By Mr. B. RUSSELL: Principles of mathematics, three hours.—By Mr. G. H. HARDY: Asymptotic relations in the theory of functions, three hours; Double limit problems, two hours.—By Mr. S. BIRTWISTLE: Thermodynamics, three hours.—By Mr. A. J. CUNNINGHAM: The principle of relativity and the mechanics of electric systems, three hours.—By Mr. G. A. CAMERON: Mechanics, three hours.

THE following university courses in mathematics are announced for the summer semester of 1914:

UNIVERSITY OF MUNICH.—By Professor F. LINDEMANN: Integral calculus, five hours; Theory of algebraic forms and invariants, four hours; Quadrature of the circle, two hours; Seminar, two hours.—By Professor A. PRINGSHEIM: Definite integrals and Fourier series, four hours; Applications of elliptic functions, three hours.—By Professor A. Voss: Analytic geometry of space, four hours; Introduction to the theory of partial differential equations, four hours; Seminar, two hours.—By Professor F. HARTOGS: Descriptive geometry, II, five hours; with exercises, four hours.—By Professor H. BRUNN: Geometric morphology, three hours.—By Dr. H. DINGLER: Introduction to higher mathematics, three hours; Exercises in plane analytic geometry, two hours.—By Dr. F. BÖHM: Elements of life insurance, four hours; Seminar in statistics, two hours.—By Dr. A. ROSENTHAL: Synthetic geometry, II, four hours; with exercises, one hour; Seminar in analysis, two hours.

UNIVERSITY OF STRASSBURG.—By Professor G. FABER: Integral calculus, four hours; Theory of numbers, two hours; Seminar.—By Professor F. SCHUR: Theoretical mechanics, four hours; Theory of ordinary differential equations, two hours; Seminar.—By Professor M. SIMON: History of mathematics in the middle ages, two hours.—By Professor J. WELLSTEIN: Elliptic and hyperelliptic functions, four hours.—By Professor R. v. MISES: Technical mechanics, three hours;

Integral equations with applications, four hours; Seminar. —By Professor P. EPSTEIN: Calculus of variations, two hours. —By Dr. A. SPEISER: Theory of assemblages, two hours; Equations of the fifth degree, one hour.

UNIVERSITY OF PARIS.—By Professor P. APPELL: Figures of equilibrium of rotating fluid bodies, two hours.—By Professor E. PICARD: Multiple integrals with applications to the theory of analytic functions of several variables, two hours. —By Professor E. GOURSAT: Ordinary and partial differential equations, two hours.—By Professor C. GUICHARD: Theoretical mechanics, two hours.—By MM. VESSIOT and MONTEL: General mathematics, two hours.—By Professor H. ANDOYER: Astronomy, two hours.—By Professor J. BOUSSINESQ: Turbulent fluid motion, two hours.—By M. A. CAHEN: The theorem of Fermat, two hours.

Conferences will be conducted by MM. LEBESGUE, DRACH, VESSIOT, MONTEL, ANDOYER, and SERVANT.

In the Ecole Normale courses in mathematics are given by Professors BOREL, CARTAN, and MM. VESSIOT, LEBESGUE, and DRACH.

PROFESSOR M. FRÉCHET, of the University of Poitiers, will give at the University of Illinois during the academic year 1914–15 a course of lectures on General analysis, including a discussion of abstract sets and functional operations. He will also conduct a seminar for students pursuing research in this field.

PROFESSOR W. C. EELLS, of Whitworth College, has been appointed instructor in mathematics and mechanics in the U. S. Naval Academy.

PROFESSOR G. B. HALSTED has resigned his position in the State Teachers College of Colorado.

PROFESSOR J. K. WHITTEMORE has resigned his assistant professorship of mathematics in Adelbert College.

DR. C. F. CRAIG, of Cornell University, has been granted a year's leave of absence for study in Europe.

BOOK catalogues: W. Engelmann, Unter den Linden 76a,

Berlin, catalogue no. 2,450 titles in exact sciences. W. Heffer and Sons, Cambridge, England, Catalogue no. 118, periodicals, 936 titles.—A. Hermann et Fils, 6 rue de la Sorbonne, Paris, 250 titles in science.—Mayer und Müller, Prinz Louis Ferdinandstrasse 2, Berlin, catalogue no. 282 about 150 mathematical titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALONSO-MISOL (F.). Analisis matematica. Madrid, 1913. 8vo. Fr. 12.00

ANDERSON (J.). Ueber eine Klasse von Untergruppen einer Abelschen G_{∞}. (*Arkiv för Matematik, Astronomi och Fysik.*) Stockholm, 1913. 8vo. 70 pp. M. 1.80

ARCHIMEDES. Geometrical solutions derived from mechanics. A treatise recently discovered and translated from the Greek by J. L. Heiberg. English version by L. G. Robinson. Chicago, Open Court Co. 28 pp. Paper. $0.30.

BÄCKLUND (A. V.). Ueber mehrdeutige Flächentransformationen. Stockholm, 1913. 4to. 88 pp. M. 4.20

BAKER (R. P.). The problem of the angle-bisectors. (Diss.) Chicago, University of Chicago Press, 1911. 4to. 106 pp. $1.25

BLOCK (H.). Sur les équations linéaires aux dérivées partielles à caractéristiques multiples. Note 4. (*Arkiv för Matematik.*) Stockholm, 1913. 8vo. 15 pp. M. 1.00

BON (F.). Ist es wahr, dass $2 \times 2 = 4$ ist? Eine experimentelle Untersuchung. 1ter Band: Von den Begriffen, den Urteilen und der Wahrheit. Leipzig, E. Reinecke, 1913. 8vo. 28 + 523 pp. M. 12.00

BOOMSTRA (W.). De orthogonale en gelijkzijdige kwadratische oppervlakken in verband met het deelingsprobleem der elliptische functies. (Thèse.) Amsterdam, Olivier, 1913. 8vo. 238 pp.

BOREL (G.). Beweisführung des Fermatschen Satzes. Rotterdam, Nijgh & VanDitmar, 1913.

CAMERON (J. F.). Ueber die Zerlegung einer Primzahl in einem komponierten Körper. (Diss.) Marburg, 1912. 8vo. 38 pp.

CLARIANA Y RICART (L.). Rapida excursion a las altas regiones del analisis matematica. (Mem. Acad.) Barcelona, 1913. 4to. 13 pp. Fr. 2.50

CRUEWELL (E. R.). Der Satz des Fermat. 2te, vermehrte Auflage. Berlin, 1913. 8vo. 4 pp. M. 2.40

ELTE (E. L.). The semiregular polytopes of the hyperspaces. (Diss.) Groningen, Gebroeders Hoitsema, 1912. 8vo. 136 pp.

GALDEANO (Z. G. DE). Sumario de mis cursos de calculo infinitesimal con arreglo al nuevo metodo de enseñanza. Zaragoza, Casanal, 1913. 8vo. 192 pp. P. 4.00

GODEAUX (L.). Sur les surfaces possédant un faisceau irrationnel de courbes hyperelliptiques de genre trois. Prague, 1913. 8vo. 6 pp. M. 0.30

HEIBERG (J. L.). See ARCHIMEDES.

HUNTINGTON (E. V.). See SYLLABUS.

KÖNIG (R.). Konforme Abbildung der Oberfläche einer räumlichen Ecke. (Habilitationsschrift.) Leipzig, 1911. 8vo. 24 pp.

KRYJANOVSKY (D.). On the maximal and minimal properties of plane figures. (Russian.) Odessa, 1913. 16mo. 4+100 pp. R. 0.50

LECAT (M.). Bibliographie du calcul des variations (1850–1913). Paris, Hermann, 1913. 8vo. 4+113 pp. Fr. 4.00

LOGIC, The algebra of. Authorized translation by L. G. Robinson. With a preface by P. E. B. Jourdain. Chicago, Open Court Co. 14+95 pp. Cloth. $1.00

MARCHAND (E.). Sur les théorèmes de Sylvester et la règle de Newton dans la théorie des équations algébriques à coefficients réelles. (Thèse, Zurich.) Neuchatel, Wolfrath et Sperlé. 16mo. 75 pp.

MARKOFF (A.). Bicentenary of the law of large numbers (1713–1913); demonstration of the second limit-theorem of the theory of probabilities by the method of moments. Supplement to the 3d Russian edition of the Calculus of probabilities. St. Petersburg, 1913. 8vo. 4+66 pp.

MONTEIL (P. L.). Théorie du point. Géométrie curviligne II. Courbes dérivées de la circonférence. Paris, Dunod et Pinat, 1913. 8vo. 124+49 pp.

PETRINI (H.). L'existence de certaines intégrales prises sur la fonction potentielle logarithmique et ses dérivés. (*Arkiv för Matematik.*) Stockholm, 1913. 8vo. 66 pp. M. 1.80

PIONCHON (J.). Notice sur la vie et les travaux de Charles Méray (1835–1911). Dijon, Marchal, 1912. 8vo. 158 pp.

ROBINSON (L. G.). See ARCHIMEDES and LOGIC.

SILBERBAUER (A.). Zur elementaren Behandlung der ebenen Schnitte von Zylinder und Kegel. (Progr.) Waidhofen a.d.T., 1913. 8vo. 9 pp.

SORIA Y MATA (A.). Génesis. Madrid, La Ciudad Lineal, 1913. 4to. 61 pp.

SPYCHER (K.). Die Schnittkurve eines gleichseitig-hyperbolischen und eines kubisch-parabolischen Cylinders. (Diss.) Bern, 1913. 8vo. 59 pp.

SYLLABUS OF MATHEMATICS. Compiled under the direction of the "Committee on the Teaching of Mathematics to Students of Engineering," by the chairman, E. V. Huntington. 2d edition. Ithaca, N. Y., Society for the Promotion of Engineering Education, 1913. 8vo. 144 pp. Cloth. $0.75

TIKHOMANDRITZKI (M.). Eléments de la théorie des intégrales abéliennes. Nouvelle édition revue, corrigée, complétée de notes et en partie refaite entièrement. St. Petersburg, Eggers, 1911. 8vo. 15+286 pp. Fr. 14.00

VENTOSA (V.). Método para la extraccion de raices de los numeros sin el auxilio de logarithmos. (Rev. Acad. Cienc. Exact.) Madrid, 1913. 8vo. 34 pp. Fr. 2.50

VIVANTI (G.). Lezioni di analisi infinitesimale. Pavia, Mattei, 1911. 8vo. 648 pp. L. 16.00

——. Esercizi di analisi infinitesimale. Pavia, Mattei, 1913. 8vo. 470 pp. L. 15.00

VOGT (H.). Solutions des exercices proposés dans les éléments de mathématiques supérieures. Paris, Vuibert & Nony, 1913. 8vo. 277 pp. Fr. 6.00

WALLNER (K.). Die Funktion $\Sigma_{i=0}^{\infty} \epsilon^{3i}/(3i)!$ und ihre Abgeleiteten. (Progr.) Rothenburg o. T., 1913. 8vo. 72 pp.

WITHERS (J. W.). Euclid's parallel postulate. Its nature, validity and place in geometrical systems. Chicago, Open Court Co. 7+192 pp. Cloth. $1.25

II. ELEMENTARY MATHEMATICS.

BARDEY (E.). Aufgabensammlung für bayrische Mittelschulen. Bearbeitet von J. Lengauer. 4te, umgeänderte Auflage. Leipzig, Teubner, 1913. 8vo. M. 2.20

BOREL (E.). Die Elemente der Mathematik. Deutsch von P. Stäckel. Heft 2: Aufgaben aus der Geometrie. Lösungen. Leipzig, Teubner, 1913. 8vo. 4+39 pp. M. 1.50

CARSON (G. ST.-L.). Essays on mathematical education. With an introduction by D. E. Smith. Boston, Ginn, 1913. 8vo. 4+139 pp. $0.75

DAVISON (C.). Problem papers. Cambridge, University Press, 1913. 8vo. 32 pp. Paper. $0.25

DENNIS (T.). An algebra for preparatory schools. Students' edition, $0.60. Teachers' edition, with interleaved answers, $1.50. Cambridge, University Press, 1913.

DEWEY (J.). See MCLELLAN (J. A.).

DICKSON (L. E.). College algebra. 2d edition. New York, Wiley, 1912. 8vo. 7+214 pp. Cloth. $1.50

GRÜNBAUM (H.). Lehr- und Aufgabenbuch der Geometrie. Planimetrie, Trigonometrie und Stereometrie. Für Maschinenbauschulen. Nach modernen Grundsätzen. Leipzig, Teubner, 1914. 8vo. 6+151 pp. M. 2.20

LENGAUER (J.). See BARDEY (E.).

MCLELLAN (J. A.) and DEWEY (J.). The psychology of number and its application to methods of teaching arithmetic. New York, 1912. 8vo. 15+309 pp. Cloth. $1.50

MARTUS (H. C. E.). Mathematische Aufgaben. Teil IV: Ergebnisse der Aufgaben von Teil III. 3te Auflage. Dresden, 1913. 8vo. 245 pp. Cloth. M. 5.50

SCHLEUSSINGER (A.). Tafel mit gekürzten Zahlenwerten zum Quadriren und Radizieren. Stuttgart, 1913. Folio. 2 pp. M. 0.60

SCHLIMBACH (A.). Faktorenzusammenstellung zur politischen Arithmetik. Frankfurt a. M., 1913. M. 4.80

STÄCKEL (P.). See BOREL (E.).

UNTERRICHT, Der mathematische, in der Schweiz. L'enseignement mathématique en Suisse. (Berichte.) 8 Hefte. Basel, 1910–12. 8vo. 749 pp. M. 16.00

VOLLPRECHT (H.). Das Rechnen eine Vorbereitung zur allgemeinen Arithmetik. Leipzig, Teubner, 1913. 8vo. M. 0.80

VORSCHLÄGE zur Vereinheitlichung der mathematischen Bezeichnungen im Schulunterricht. (Schriften des deutschen Ausschusses. Heft 17.) Leipzig, Teubner, 1913. 8vo. 4+14 pp.

WALSH (J. H.). Introductory algebra. Boston, Heath, 1912. 12mo. Cloth. $0.50

WELLS (W.) and HART (W. W.). New high school algebra. Boston, Heath, 1913. 12mo. 431 pp. Cloth. $1.20

III. APPLIED MATHEMATICS.

ADAMS (H.). Practical surveying and elementary geodesy. London, Macmillan, 1913. 8vo. 12+276 pp. 4s. 6d.

AKESSON (O. A.). Formler och Exempel till sfärisk Astronomi. Lund, 1913. 3vo. 4+102 pp. M. 3.00.

BALDWIN (A. L.). The California-Washington arc of primary triangulation. (Coast and Geodetic Survey.) Washington, 1913. 4to. 78 pp. $1.25

BOHLIN (K.). Sur le développement des intégrales du problème des trois corps. Partie 1: Rayon vecteur par rapport au centre de gravité binaire. (*Arkiv för Matematik.*) Stockholm, 1913. 8vo. 28 pp. M. 1.00

BOUASSE (H.). Cours de physique. Tome I: Cours de mécanique physique. 2e édition, complètement transformée et considérablement augmentée. Paris, Delagrave, 1912. 8vo. 684 pp. Broché. Fr. 20.00

BOULVIN (J.). Cours de mécanique appliquée aux machines. (Ouvrage ayant obtenu un prix de l'Académie des Sciences.) 5e volume: Machines à vapeur et turbines à vapeur. 3e édition, revue et corrigée. Paris, Geisler, 1912. 8vo. 612 pp. Fr. 15.00

——. Cours de mécanique appliquée aux machines. 6e volume, 2e édition, divisée en 2 parties. 1re partie: Locomotives. Paris, Geisler, 1912. 8vo. 380 pp. Fr. 7.50

BOWIE (W.). Determination of time, longitude, latitude and azimuth. 5th edition. (Coast and geodetic survey.) Washington, 1913. 4to. 177 pp. $2.50

BREWSTER (G. W.) and WAGSTAFF (C. J. L.). A school statics. London, Heffer, 1913. 8vo. 256 pp. 3s.

BUCHANAN (J. Y.). Scientific papers. Volume 1. Cambridge, University Press, 1912. $3.25

CAPITO (C. A. A.). A textbook of mathematics and mechanics. For the use of students qualifying for science and technical examinations. London, Griffin, 1913. 8vo. 15+398 pp. Cloth. 12s. 6d.

CHOMÉ (F.). Cours de géométrie descriptive. Livre 1er. 5e édition, revue et augmentée. Paris, Gauthier-Villars, 1912. 22+211 pp. Fr. 10.00

CHRISTIE (C. V.). Electrical engineering; the theory and characteristics of electrical machinery. New York, McGraw-Hill, 1913. 8vo. 417 pp. Cloth. $4.00

DUNLOP (H. C.) and JACKSON (C. S.). Slide rule notes. London, Longmans, 1913. 8vo. 136 pp. 2s. 6d.

ENCYCLOPÉDIE ÉLECTROTECHNIQUE. Par un comité d'ingénieurs spécialistes. Fascicule 12: Dynamos à courant alternatif; calcul, construction. Fascicule 15: Transformateurs. Fascicule 16: Transformation des courants. Fascicule 23: Iæs isolants employés dans la pratique. Fascicule 28: Limiteurs de tension; parafoudres. Fascicule 29: Appareillage d'interruption. Fascicule 30: Lampes à incandescence. Fascicule 32: Moteurs à courant continu. Fascicule 49: Traction électrique à courant continu. Fascicule 50: Traction électrique à courant alternatif. Paris, Geisler, 1913. 8vo. Chaque fascicule, Fr. 2.50

ESCHER (W.). Ueber die spezifische Wärme von elektrolytischem Wasserstoff und ihre Beziehungen zur kinetischen Theorie zweiatomiger Gase, nebst einer theoretischen Bestimmung des mechanischen Wärmeäquivalentes. (Diss.) Marburg, 1912. 8vo. 77 pp.

GEIGER (K.). Das Biot-Savartsche Gesetz. (Progr.) Landshut, 1912. 8vo. 15 pp.

GREVE (B. DE) en RUTGERS (J. G.). Leerboek der Mechanica. Amsterdam, Scheltema & Holkema, 1912. 8vo. 191 pp.

HINKS (A. R.). Maps and surveys. Cambridge, University Press, 1913. 8vo. 16+206 pp. $2.00

JACKSON (C. S.). See DUNLOP (H. C.).

JAMIESON (A.). A text-book of applied mechanics and mechanical engineering. Volume 5. 8th edition, revised and enlarged. London, Griffin, 1913. 8vo. 546 pp. 7s. 6d.

JORDAN (L. C.). The practical railway spiral. London, Constable, 1913. 8vo. 8+155 pp. Leather. 6s.

JULLY (A.). Menuisier en bâtiment. 1er cahier: Géométrie, dessin technique et construction. Paris, Geisler, 1913. 8vo. Fr. 1.25

——. Modeleur-mécanicien. 1er cahier: Géométrie, etc. Paris, Geisler, 1913. 8vo. Fr. 1.25

LEWIS (G. N.). See WILSON (E. B.).

MAENNEL (F.). Wie weit lässt sich das Verhältnis der Energiemengen in zwei lose gekoppelten elektrischen Schwingungskreisen durch Spannungsmessungen ermitteln? (Diss.) Halle, 1912. 4to. 44 pp.

NORRIS (H. H.). An introduction to the study of electrical engineering. 2d edition, revised. New York, Wiley, 1912. 8vo. 5+418 pp. Cloth. $2.50

OSEEN (C. W.). Ueber die elektromagnetischen Schwingungen an dünnen Ringen. (*Arkiv för Matematik.*) Stockholm, 1913. 8vo. 30 pp. M. 1.00

PALMER (C. I.). Practical mathematics. In 4 parts. Part III: Algebra with applications, 189 pp. Part IV: Trigonometry and logarithms, 160 pp. New York, McGraw-Hill, 1913. 12mo. $1.50

PILIZOTTI (K.). Geometrographische Untersuchung einiger photogrammetrischer Probleme. (Progr.) Wien, 1912. 8vo. 10 pp.

POINCARÉ (H.). Die neue Mechanik. 2te Auflage. Leipzig, Teubner, 1913. 8vo. 22 pp. M. 0.60

PUTNOKY (L. v.). Ueber Potentialsprünge an der Grenzfläche zwischen Flüssigkeit und Gas. (Diss.) Karlsruhe, 1913. 8vo. 66 pp.

RÖMER (V.). Zur Theorie der Wärmestrahlung. (Progr.) Wien, 1912. 8vo. 54 pp.

RUDEL (E.). Die Orientierung photogrammetrischer Aufnahmen bei vertikaler Bildebene unter Benutzung magnetischer Azimute. (Progr.) Ludwigshafen a. R., 1912. 8vo. 40 pp.

RUTGERS (J. G.). See GREVE (B. DE).

SCHREBER (K.). Hervorragende Leistungen der Technik. 1ter Band: Technische Physik. Leipzig, Teubner, 1913. 8vo. 4+207 pp. M. 3.00

SCHWEER (W.). Untersuchungen über gekoppelte unsymmetrische Schwingungskreise. (Diss.) Halle, 1912. 4to. 19 pp.

SIBLEY (F. H.). Pure mechanism. New York, Holt, 1913. 8vo. 9+285 pp. $3.00

SUMPF (K.). Grundriss der Physik. Ausgabe A. Herausgegeben von Hartenstein und Pabst. Hildesheim, Lax, 1913. M. 3.80

WAGSTAFF (C. J. L.). See BREWSTER (G. W.).

WALKER (J.). Introduction to physical chemistry. 7th edition. New York, Macmillan, 1913. 8vo. 12+421 pp. Cloth. $2.60

WIGHT (J. T.). Elementary graphic statics. London, Whittaker, 1913. 8vo. 240 pp. 4s.

WILDENBROTHER (I.). Bewegung eines Punktes unter dem Zwang von totalen linearen Differentialgleichungen. (Progr.) Landau i. Pfalz., 1912. 8vo. 31 pp.

WILSON (D. T.). Tables for the computation of the Jupiter perturbations of the group of small planets whose mean daily motions are in the neighborhood of 750″. Stockholm, 1912. 4to. 58 pp.

WILSON (E. B.) and LEWIS (G. N.). The space-time manifold of relativity. The non-euclidean geometry of mechanics and electromagnetics. (*Proceedings of the American Academy of Science.*) Boston, 1912. 8vo. 121 pp. $2.25

WIMPERIS (H. E.). The principles of the application of power to road transport. London, Constable, 1913. 8vo. 146 pp. 4s. 6d.